

Some Extensions of Witt's Theorem

Huajun Huang

ABSTRACT

We extend Witt's theorem to several kinds of simultaneous isometries of subspaces. We determine sufficient and necessary conditions for the extension of an isometry of subspaces $\phi : E \rightarrow E'$ to an isometry $\phi_V : V \rightarrow V'$ that also sends a given subspace to another, or a given self-dual flag to another, or a Witt's decomposition to another and a special self-dual flag to another. We also determine sufficient and necessary conditions for the isometry of generic flags or the simultaneous isometry of (subspace, self-dual flag) pairs.

1. Introduction

Let \mathbb{F} be a field with $\text{char}(\mathbb{F}) \neq 2$. Let (V, b) be a finite dimensional metric space over \mathbb{F} with a nonsingular symmetric bilinear form $b : V \times V \rightarrow \mathbb{F}$. The classical Witt's theorem [31] states that every isometry between two subspaces of V can be extended to an isometry of the whole space V . Witt's theorem has been widely extended in different context, such as on the fields of characteristic 2 [1, 2, 6, 24, 25], on various types of local rings [7–9, 15, 16, 18–20, 22, 29, 30], on the spaces of countable dimension [3–5, 17], on the noncommutative rings [21, 23, 26], and on some other situations [13, 14, 28].

The present paper extends the classical Witt's theorem in another direction. Let (V, b) and (V', b') be isometric nonsingular symmetric metric spaces. Let $E \subset V$ and $E' \subset V'$ (resp. $A \subset V$ and $A' \subset V'$) denote isometric subspaces. Let \mathcal{V} (resp. \mathcal{V}') denote a flag of V (resp. V'), that is, an ascending chain of subspaces of V . We provide:

- i) A sufficient and necessary condition for an isometry $\phi : E \rightarrow E'$ to be extended to an isometry of the whole space $\phi_V : V \rightarrow V'$ that satisfies one of the following:
 - (a) ϕ_V sends a given subspace $A \subset V$ to another given subspace $A' \subset V'$ (Theorem 5.3).
 - (b) ϕ_V sends a given self-dual flag \mathcal{V} to another given self-dual flag \mathcal{V}' (Theorem 3.2).
 - (c) ϕ_V sends a given Witt's decomposition $V = V^+ \oplus \widehat{V} \oplus V^-$ of V to another given Witt's decomposition $V' = V'^+ \oplus \widehat{V'} \oplus V'^-$ of V' , and sends a given self-dual flag \mathcal{V} “compatible” with V^+ to another given self-dual flag \mathcal{V}' “compatible” with V'^+ (Theorem 4.4).
- ii) A sufficient and necessary condition for the existence of an isometry $\phi_V : V \rightarrow V'$ that satisfies one of the following:
 - (a) ϕ_V sends a given generic flag \mathcal{V} to another given generic flag \mathcal{V}' (Theorem 6.1).
 - (b) ϕ_V sends a given subspace $E \subset V$ to a given subspace $E' \subset V'$, and a given self-dual flag \mathcal{V} of V to a given self-dual flag \mathcal{V}' of V' (Theorem 6.2).

The work has the merit of embracing all classical forms— analogous results hold when b and b' are alternating, Hermitian, or skew-Hermitian forms, and the proofs are similar. It is interesting to extend the results to other algebraic structures like skew-fields, local rings, or to characteristic 2 situations.

2. Preliminary

Let (V, b) and (V', b') be isometric finite dimensional symmetric metric spaces over \mathbb{F} , *either singular or nonsingular* (in this section only). This means that there is a linear bijection $\phi_V : V \rightarrow V'$ such that $b(\mathbf{u}, \mathbf{v}) = b'(\phi_V(\mathbf{u}), \phi_V(\mathbf{v}))$ for all $\mathbf{u}, \mathbf{v} \in V$. We call ϕ_V an *isometry*.

We adopt some conventional notations from [27]. The metric space (V, b) is *anisotropic* if $b(\mathbf{v}, \mathbf{v}) \neq 0$ for all nonzero vector $\mathbf{v} \in V$. It is *totally isotropic* if $b(\mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{v} \in V$. Let A^\perp denote the orthogonal complement of A . The *radical* of a metric space (or metric subspace) A is $A^\perp \cap A$, or simply A^\perp if A is the whole space. Obviously, a radical is totally isotropic. Let $A \oplus B$ denote the direct sum, and $A \odot B$ denote the *orthogonal* direct sum, of subspaces A and B of a metric space. Let $A - B := \{\mathbf{v} \in A \mid \mathbf{v} \notin B\}$ (which is unlike the notation of $A + B$). Write $A \approx A'$ when A and A' are isometric with respect to the corresponding metrics, and $A \overset{\phi}{\approx} A'$ when A and A' are isometric via an isometry ϕ (the domain of ϕ may be larger than A).

The following two lemmas are handy in the later computations.

LEMMA 2.1. Let A, B, C, E be subspaces of the metric space (V, b) .

- (1) $\dim(A + B) + \dim(A \cap B) = \dim A + \dim B$.
- (2) If V is nonsingular, then $\dim A + \dim A^\perp = \dim V$.
- (3) $(A + B)^\perp = A^\perp \cap B^\perp$ and $(A \cap B)^\perp = A^\perp + B^\perp$.
- (4) $A \subset B$ if and only if $A^\perp \supset B^\perp$.
- (5) If $B \subset C$, then $(A + B) \cap C = A \cap C + B$.
- (6) If $A \subset E$, then $A \cap B = A \cap E \cap B$.
- (7) If $A \subset E$ and $B \subset E$, then $A \cap (B + C) = A \cap (B + C \cap E)$.
- (8) $\dim A - \dim(A \cap B^\perp) = \dim B - \dim(B \cap A^\perp)$.

LEMMA 2.2. Suppose $A, B \subset V$ and $A', B' \subset V'$ are metric subspaces.

- (1) If $A \overset{\phi}{\approx} A'$, then every isometry $\phi_V : V \rightarrow V'$ extended from ϕ satisfies $A^\perp \overset{\phi_V}{\approx} A'^\perp$.
- (2) If $A \overset{\phi}{\approx} A'$ and $B \overset{\phi}{\approx} B'$, then $A \cap B \overset{\phi}{\approx} A' \cap B'$ and $A + B \overset{\phi}{\approx} A' + B'$.

All proofs are trivial except for that of Lemma 2.1(8). Select a basis of $A \cap B^\perp$ and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_{\dim A}\}$ of A . Select a basis of $B \cap A^\perp$ and extend it to a basis $\{\mathbf{y}_1, \dots, \mathbf{y}_{\dim B}\}$ of B . Then Lemma 2.1(8) comes from the equalities:

$$\dim A - \dim(A \cap B^\perp) = \text{rank} [b(\mathbf{x}_i, \mathbf{y}_j)]_{\dim A \times \dim B} = \dim B - \dim(B \cap A^\perp).$$

Given linear maps $\mu : A \rightarrow A'$ and $\psi : B \rightarrow B'$ where $A, B \subset V$ and $A', B' \subset V'$, we define $\mu \oplus \psi$ naturally whenever $A \oplus B$ and $A' \oplus B'$ are well-defined. Likewise for $\mu \odot \psi$. More generally, if μ and ψ agree on the intersection of domains $\text{Dom}(\mu) \cap \text{Dom}(\psi) = A \cap B$, then there is a unique linear map ρ from $A + B$ to $A' + B'$ that naturally extends both μ and ψ . Call ρ the *combination* of μ and ψ . A quick observation shows that if both μ and ψ are linear bijections, then so is the combination of μ and ψ . Nevertheless, if both μ and ψ are isometries, then the combination of μ and ψ may not necessarily be an isometry.

Witt's theorem can be slightly extended to include singular cases.

LEMMA 2.3. Every isometry $\phi : E \rightarrow E'$ from subspace E of V to subspace E' of V' with $\phi(E \cap V^\perp) = E' \cap V'^\perp$ can be extended to an isometry from V to V' .

Proof. Select subspace \tilde{E} such that $E = (E \cap V^\perp) \odot \tilde{E}$. Extend \tilde{E} to \tilde{V} such that $V = V^\perp \odot \tilde{V}$, where V^\perp is the radical of V . Denote $\tilde{E}' := \phi(\tilde{E})$. Then

$$E' = \phi(E \cap V^\perp) \odot \phi(\tilde{E}) = (E' \cap V'^\perp) \odot \tilde{E}'.$$

Extend \tilde{E}' to \tilde{V}' such that $V' = V'^\perp \odot \tilde{V}'$. From $V \approx V'$ we get $\tilde{V} \approx V/V^\perp \approx V'/V'^\perp \approx \tilde{V}'$, where \tilde{V} and \tilde{V}' are nonsingular. By Witt's theorem (with respect to V and V') the isometry $\phi|_{\tilde{E}}$ can be extended to an isometry $\phi_{\tilde{V}} : \tilde{V} \rightarrow \tilde{V}'$. Moreover, since $\phi(E \cap V^\perp) = E' \cap V'^\perp$ and $\dim V^\perp = \dim V'^\perp$, the isometry $\phi|_{E \cap V^\perp}$ can be extended to an isometry of the totally isotropic subspaces $\phi_{V^\perp} : V^\perp \rightarrow V'^\perp$. Thus $\phi_V := \phi_{V^\perp} \odot \phi_{\tilde{V}}$ is an isometry from V to V' that extends ϕ . \square

There is a parallel result with no initial isometry provided.

LEMMA 2.4. Let $E \subset V$ and $E' \subset V'$ such that $E \approx E'$ and $\dim(E \cap V^\perp) = \dim(E' \cap V'^\perp)$. Then there exists an isometry $\phi_V : V \rightarrow V'$ with $\phi_V(E) = E'$.

Proof. Construct a linear bijection (i.e. an isometry) $\phi_0 : E \cap V^\perp \rightarrow E' \cap V'^\perp$. Then

$$\phi_0((E \cap V^\perp) \cap E^\perp) = \phi_0(E \cap V^\perp) = E' \cap V'^\perp = (E' \cap V'^\perp) \cap E'^\perp.$$

Apply Lemma 2.3 on isometric metric spaces $E \approx E'$ and the isometry ϕ_0 . Then ϕ_0 can be extended to an isometry $\phi_1 : E \rightarrow E'$ with $\phi_1(E \cap V^\perp) = E' \cap V'^\perp$. Apply Lemma 2.3 again on $V \approx V'$ and the isometry ϕ_1 . Then ϕ_1 can be extended to an isometry $\phi_V : V \rightarrow V'$. Clearly $\phi_V(E) = E'$. \square

Lemma 2.3 and Lemma 2.4 lead to the following results.

COROLLARY 2.5. Suppose $V \approx V'$ are nonsingular. Let $E, A \subset V$ and $E', A' \subset V'$ satisfy that $E \perp A$, $E' \perp A'$, $A \approx A'$, and $\phi : E \rightarrow E'$ is an isometry with $\phi(E \cap A) = E' \cap A'$. Then ϕ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends A to A' .

Proof. Notice that $(A^\perp)^\perp = A$ and $(A'^\perp)^\perp = A'$. Apply Lemma 2.3 on subspace E of A^\perp and subspace E' of A'^\perp . Then the isometry ϕ can be extended to an isometry $\phi_1 : A^\perp \rightarrow A'^\perp$, which can be further extended to an isometry $\phi_V : V \rightarrow V'$ by Witt's theorem. Then ϕ_V sends A to A' . \square

COROLLARY 2.6. Suppose $V \approx V'$ are nonsingular. Let $E, A \subset V$ and $E', A' \subset V'$ satisfy that $E \perp A$, $E' \perp A'$, $E \approx E'$, $A \approx A'$, and $\dim(E \cap A) = \dim(E' \cap A')$. Then there exists an isometry $\phi_V : V \rightarrow V'$ such that $E \overset{\phi_V}{\approx} E'$ and $A \overset{\phi_V}{\approx} A'$.

Proof. Again $(A^\perp)^\perp = A$ and $(A'^\perp)^\perp = A'$. Applying Lemma 2.4 on subspace E of A^\perp and subspace E' of A'^\perp , we get an isometry $\phi_1 : A^\perp \rightarrow A'^\perp$ that sends E to E' . It can be extended to an isometry $\phi_V : V \rightarrow V'$ by Witt's theorem. Then $E \overset{\phi_V}{\approx} E'$ and $A \overset{\phi_V}{\approx} A'$. \square

3. Simultaneous isometry of (subspace, self-dual flag) pairs

From now on, let $(V, b) \approx (V', b')$ be *nonsingular* isometric symmetric metric spaces over \mathbb{F} . In this section, we determine when an isometry $\phi : E \rightarrow E'$ between subspaces $E \subset V$ and $E' \subset V'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends a given self-dual flag of V to another given self-dual flag of V' .

Let $\mathbf{0}$ (resp. $\mathbf{0}'$) denote the zero vector in V (resp. V').

A *flag* \mathcal{V} of V is a chain of embedding subspaces of V :

$$\mathcal{V} := \{V_0 \subset V_1 \subset \cdots \subset V_k\}. \quad (1)$$

For convenience, we set V_0 the zero space and V_k the whole space throughout the paper, and write $\mathcal{V} = \{V_i\}_{i=0,\dots,k}$. The flag \mathcal{V} is called a *self-dual flag* if

$$\{V_k^\perp \subset V_{k-1}^\perp \subset \dots \subset V_0^\perp\} = \{V_0 \subset V_1 \subset \dots \subset V_k\}. \quad (2)$$

When \mathcal{V} is a self-dual flag, the subspaces in the “first half part” of \mathcal{V} are totally isotropic.

Let $\mathcal{V} = \{V_i\}_{i=0,\dots,k}$ (resp. $\mathcal{V}' = \{V'_i\}_{i=0,\dots,k'}$) be a flag of V (resp. V'). Given a linear map $\phi_V : V \rightarrow V'$, we write $\phi_V(\mathcal{V}) = \mathcal{V}'$ if $k = k'$ and $\phi_V(V_i) = V'_i$ for $i = 0, \dots, k$. Denote $\mathcal{V} \approx \mathcal{V}'$ or $\mathcal{V} \stackrel{\phi_V}{\approx} \mathcal{V}'$ (when ϕ_V is given) if there is an isometry $\phi_V : V \rightarrow V'$ such that $\phi_V(\mathcal{V}) = \mathcal{V}'$.

LEMMA 3.1. Suppose \mathcal{V} and \mathcal{V}' defined above are self-dual flags. Then $\mathcal{V} \approx \mathcal{V}'$ if and only if $k = k'$ and $\dim V_i = \dim V'_i$ for $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$.

Proof. Only the sufficient part is nontrivial. By the sufficient conditions in the lemma, we can construct a linear bijection $\phi : V_{\lfloor \frac{k}{2} \rfloor} \rightarrow V'_{\lfloor \frac{k}{2} \rfloor}$ such that $\phi(V_i) = V'_i$ for $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$. Then ϕ is an isometry of totally isotropic subspaces. By Witt's theorem ϕ can be extended to an isometry $\phi_V : V \rightarrow V'$. Clearly $\mathcal{V} \stackrel{\phi_V}{\approx} \mathcal{V}'$. \square

The criteria for the isometry of two generic flags will be determined in Section 6.

Let \mathcal{V} and \mathcal{V}' be generic flags with $k = k'$. Given $E \subset V$ and $E' \subset V'$, we denote

$$E_i := E \cap V_i, \quad E'_i := E' \cap V'_i, \quad \text{for } i = 0, \dots, k. \quad (3)$$

THEOREM 3.2. Let $\mathcal{V} := \{V_i\}_{i=0,\dots,k}$ and $\mathcal{V}' := \{V'_i\}_{i=0,\dots,k}$ be isometric self-dual flags of V and V' respectively. Then an isometry $\phi : E \rightarrow E'$ from $E \subset V$ to $E' \subset V'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends \mathcal{V} to \mathcal{V}' , if and only if $\phi(E_i) = E'_i$ for $i = 0, \dots, k$.

The result plays a key role in the author's classification of Borel subgroup orbits of classical symmetric subgroups on multiplicity-free flag varieties ([10–12], preprints). The cases of $k \leq 2$ may be viewed as special cases of $k = 3$ by adding certain repeated subspace(s) on self-dual flags of $k \leq 2$. Surprisingly, $k = 3$ implies the general cases.

LEMMA 3.3. For $k = 3$, Theorem 3.2 is true.

Proof of Theorem 3.2 by Lemma 3.3. It suffices to prove the sufficient part.

i) Consider the isometry $\phi : E \rightarrow E'$ and the self-dual flags

$$\{\{\mathbf{0}\} \subset V_1 \subset V_{k-1} \subset V\} \quad \text{and} \quad \{\{\mathbf{0}'\} \subset V'_1 \subset V'_{k-1} \subset V'\}.$$

Since $\phi(E_i) = E'_i$ for $i = 0, 1, k-1, k$, by Lemma 3.3 the isometry ϕ can be extended to an isometry $\phi_1 : V \rightarrow V'$ that sends V_1 to V'_1 . The restricted map $\phi_1|_{E+V_1}$ is an isometry from $E + V_1$ to $E' + V'_1$, which sends E to E' and V_1 to V'_1 respectively.

ii) Consider the isometry $\phi_1|_{E+V_1}$ and the self-dual flags

$$\{\{\mathbf{0}\} \subset V_2 \subset V_{k-2} \subset V\} \quad \text{and} \quad \{\{\mathbf{0}'\} \subset V'_2 \subset V'_{k-2} \subset V'\}.$$

Clearly $V_1 \subset V_i$ and $V'_1 \subset V'_i$ for $i = 2, k-2$. By Lemma 2.1(5),

$$\begin{aligned} \phi_1|_{E+V_1}((E + V_1) \cap V_i) &= \phi_1(E \cap V_i + V_1) = \phi(E_i) + \phi_1(V_1) \\ &= E'_i + V'_1 = (E' + V'_1) \cap V'_i \quad \text{for } i = 2, k-2. \end{aligned}$$

By Lemma 3.3, the map $\phi_1|_{E+V_1}$ can be extended to an isometry $\phi_2 : V \rightarrow V'$ that sends V_2 to V'_2 . The restricted map $\phi_2|_{E+V_2}$ is an isometry from $E + V_2$ to $E' + V'_2$, which sends E to E' , V_1 to V'_1 , and V_2 to V'_2 respectively.

- iii) Repeat the above process. Eventually, $\phi : E \rightarrow E'$ can be extended to an isometry $\phi_{\lfloor \frac{k}{2} \rfloor} : V \rightarrow V'$ that sends V_i to V'_i for $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$. Then $\phi_V := \phi_{\lfloor \frac{k}{2} \rfloor}$ is an extension of ϕ , with $\phi_V(V_i) = V'_i$ and $\phi_V(V_{k-i}) = \phi_V(V_i^\perp) = \phi_V(V_i)^\perp = V'_{k-i}$ for $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$. This completes the proof. \square

Proof of Lemma 3.3. Only the sufficient part is nontrivial. Suppose

$$\mathcal{V} := \{V_i\}_{i=0, \dots, 3} \quad \text{and} \quad \mathcal{V}' := \{V'_i\}_{i=0, \dots, 3}$$

are isometric self-dual flags of V and V' respectively, and the isometry $\phi : E \rightarrow E'$ satisfies that $\phi(E_i) = E'_i$ for $i = 0, 1, 2, 3$. We extend ϕ to an isometry $\phi_1 : E + V_1 \rightarrow E' + V'_1$ by the self-explained diagram:

$$\begin{array}{ccccc} E_1 & \hookrightarrow & E_1 + (V_1 \cap E^\perp) & \hookrightarrow & V_1 \\ \phi|_{V_1} \downarrow & & \phi_0|_{V_1} \downarrow & & \phi_1|_{V_1} \downarrow \\ E'_1 & \hookrightarrow & E'_1 + (V'_1 \cap E'^\perp) & \hookrightarrow & V'_1 \end{array}$$

The total isotropies of V_1 and V'_1 make the constructions of isometries much easier.

- i) Extend $\phi : E \rightarrow E'$ to an isometry $\phi_0 : E + (V_1 \cap E^\perp) \rightarrow E' + (V'_1 \cap E'^\perp)$.
By Lemma 2.1,

$$\begin{aligned} \dim(V_1 \cap E^\perp) &= \dim V_1 + \dim E^\perp - \dim(V_1 + E^\perp) \\ &= \dim V_1 + (\dim V - \dim E) - [\dim V - \dim(V_1 + E^\perp)] \\ &= \dim V_1 - \dim E + \dim(V_1^\perp \cap E^{\perp\perp}) \\ &= \dim V_1 - \dim E + \dim E_2. \end{aligned}$$

Similar computation on $\dim(V'_1 \cap E'^\perp)$ shows that $\dim(V_1 \cap E^\perp) = \dim(V'_1 \cap E'^\perp)$. Then $V_1 \cap E^\perp \approx V'_1 \cap E'^\perp$ since both spaces are totally isotropic. Notice that

$$E \cap (V_1 \cap E^\perp) = E_1 \cap E^\perp, \quad E' \cap (V'_1 \cap E'^\perp) = E'_1 \cap E'^\perp, \quad \phi(E_1 \cap E^\perp) = E'_1 \cap E'^\perp.$$

Applying Corollary 2.5 on $\phi : E \rightarrow E'$, $A := V_1 \cap E^\perp$ and $A' := V'_1 \cap E'^\perp$, we get an isometric extension of ϕ that sends $V_1 \cap E^\perp$ to $V'_1 \cap E'^\perp$. That is, ϕ can be extended to an isometry

$$\phi_0 : E + (V_1 \cap E^\perp) \rightarrow E' + (V'_1 \cap E'^\perp).$$

- ii) Extend $\phi_0 : E + (V_1 \cap E^\perp) \rightarrow E' + (V'_1 \cap E'^\perp)$ to an isometry $\phi_1 : E + V_1 \rightarrow E' + V'_1$.
In V , select \tilde{E} such that $E = E_2 \oplus \tilde{E}$. Select $\overline{E} \subset E_1$ and expand it to $\overline{V} \subset V_1$ such that

$$E_1 = (E_1 \cap E^\perp) \odot \overline{E}, \quad V_1 = (V_1 \cap E^\perp) \odot \overline{V}. \quad (4)$$

Then by Lemma 2.1(8),

$$\dim \overline{V} = \dim V_1 - \dim(V_1 \cap E^\perp) = \dim E - \dim(E \cap V_1^\perp) = \dim \tilde{E}. \quad (5)$$

The Riesz representation $\tau_b : \overline{V} \rightarrow (\tilde{E})^*$, defined by

$$\tau_b(\mathbf{v})(\mathbf{e}) := b(\mathbf{e}, \mathbf{v}) \quad \text{for } \mathbf{v} \in \overline{V}, \quad \mathbf{e} \in \tilde{E},$$

is injective since $\overline{V} \subset V_1 = V_2^\perp \subset E_2^\perp$ and so

$$\ker \tau_b = \overline{V} \cap \tilde{E}^\perp = \overline{V} \cap E_2^\perp \cap \tilde{E}^\perp = \overline{V} \cap (E_2 + \tilde{E})^\perp = \overline{V} \cap E^\perp = \{\mathbf{0}\}.$$

Hence τ_b is an isomorphism by (5).

In V' , let $\tilde{E}' := \phi_0(\tilde{E})$ so that $E' = E'_2 \oplus \tilde{E}'$; Let $\overline{E}' := \phi_0(\overline{E})$. By (4),

$$E'_1 = \phi_0(E_1 \cap E^\perp) \odot \phi_0(\overline{E}) = (E'_1 \cap E'^\perp) \odot \overline{E}'.$$

Expand \overline{E}' to $\overline{V}' \subset V'_1$ such that $V'_1 = (V'_1 \cap E'^\perp) \odot \overline{V}'$. Likewise, $\dim \overline{V}' = \dim \tilde{E}'$. The Riesz representation $\tau_{b'} : \overline{V}' \rightarrow (\tilde{E}')^*$, defined by

$$\tau_{b'}(\mathbf{v}')(\mathbf{e}') := b'(\mathbf{e}', \mathbf{v}') \quad \text{for } \mathbf{v}' \in \overline{V}', \mathbf{e}' \in \tilde{E}',$$

is also an isomorphism.

We now construct an isometry $\phi_\beta : \overline{V} \rightarrow \overline{V}'$ such that ϕ_0 and ϕ_β agree on the intersection of domains, and the combination ϕ_1 of ϕ_0 and ϕ_β is an isometry.

Let ϕ_β be induced by the commutative diagram:

$$\begin{array}{ccc} \overline{V} & \xrightarrow{\tau_b} & (\tilde{E})^* \\ \phi_\beta \downarrow & & \downarrow (\phi_0^{-1}|_{\tilde{E}'})^* \\ \overline{V}' & \xrightarrow{\tau_{b'}} & (\tilde{E}')^* \end{array}$$

that is, $\phi_\beta : \overline{V} \rightarrow \overline{V}'$ is uniquely determined by

$$b'(\phi_0(\mathbf{e}), \phi_\beta(\mathbf{v})) = b(\mathbf{e}, \mathbf{v}) \quad \text{for } \mathbf{v} \in \overline{V} \text{ and } \mathbf{e} \in \tilde{E}. \quad (6)$$

Then $\phi_\beta = \tau_{b'}^{-1} \circ (\phi_0^{-1}|_{\tilde{E}'})^* \circ \tau_b$ is an isometry between totally isotropic spaces \overline{V} and \overline{V}' . By Lemma 2.1(5) and (4),

$$\text{Dom}(\phi_0) \cap V_1 = [E + (V_1 \cap E^\perp)] \cap V_1 = E_1 + (V_1 \cap E^\perp) = \overline{E} \odot (V_1 \cap E^\perp).$$

So by (4), $\overline{V} \subset V_1$ and thus

$$\text{Dom}(\phi_0) \cap \text{Dom}(\phi_\beta) = \text{Dom}(\phi_0) \cap V_1 \cap \overline{V} = \overline{E}.$$

Given $\overline{\mathbf{e}} \in \overline{E}$, we have $\phi_0(\overline{\mathbf{e}}) \in \overline{E}' \subset \text{Im}(\phi_\beta)$, and by (6)

$$b'(\phi_0(\mathbf{e}), \phi_\beta(\overline{\mathbf{e}})) = b(\mathbf{e}, \overline{\mathbf{e}}) = b'(\phi_0(\mathbf{e}), \phi_0(\overline{\mathbf{e}})) \quad \text{for all } \mathbf{e} \in \tilde{E}.$$

So the isometries ϕ_0 and ϕ_β agree on \overline{E} , the intersection of domains. Therefore, the combination ϕ_1 of ϕ_0 and ϕ_β is a linear bijection and

$$\text{Dom}(\phi_1) = \text{Dom}(\phi_0) + \text{Dom}(\phi_\beta) = E + (V_1 \cap E^\perp) + \overline{V} = E + V_1.$$

Likewise, $\text{Im}(\phi_1) = E' + V'_1$.

To prove that ϕ_1 is an isometry, it remains to show that

$$b'(\phi_1(\mathbf{u}), \phi_1(\mathbf{v})) = b(\mathbf{u}, \mathbf{v}) \quad (7)$$

for $\mathbf{u} \in \text{Dom}(\phi_0)$ and $\mathbf{v} \in \text{Dom}(\phi_\beta)$. By $\tilde{E} \cap [E_2 + (V_1 \cap E^\perp)] \subset \tilde{E} \cap V_2 = \{\mathbf{0}\}$,

$$\text{Dom}(\phi_0) = (\tilde{E} \oplus E_2) + (V_1 \cap E^\perp) = \tilde{E} \oplus [E_2 + (V_1 \cap E^\perp)]. \quad (8)$$

By $E_2 + (V_1 \cap E^\perp) \subset V_2 \subset \overline{V}^\perp = \text{Dom}(\phi_\beta)^\perp$ and

$$\phi_1(E_2 + (V_1 \cap E^\perp)) = E'_2 + (V'_1 \cap E'^\perp) \subset V'_2 \subset \overline{V}'^\perp = \text{Im}(\phi_\beta)^\perp,$$

it suffices to prove (7) for $\mathbf{u} \in \tilde{E}$ and $\mathbf{v} \in \text{Dom}(\phi_\beta) = \overline{V}$. This is exactly (6).

Therefore, $\phi_1 : E + V_1 \rightarrow E' + V'_1$ is an isometry, where

$$\phi_1(V_1) = \phi_0(V_1 \cap E^\perp) \odot \phi_\beta(\overline{V}) = (V'_1 \cap E'^\perp) \odot \overline{V}' = V'_1.$$

Finally, by Witt's theorem, ϕ_1 can be extended to an isometry $\phi_V : V \rightarrow V'$. Then $\phi_V|_E = \phi$, $\phi_V(V_1) = \phi_1(V_1) = V'_1$, and $\phi_V(V_2) = \phi_V(V_1^\perp) = \phi_V(V_1)^\perp = V'_2$. \square

4. Applications of Theorem 3.2 to Witt's Decomposition

In this section, we apply Theorem 3.2 to solve the following problem: When can an isometry of subspaces be extended to an isometry of whole spaces that preserves the corresponding Witt's decompositions and the corresponding self-dual flags "compatible" with the Witt's decompositions?

LEMMA 4.1. Let $V = A \oplus B$ and $V' = A' \oplus B'$ be vector space decompositions such that $\dim A = \dim A'$ and $\dim B = \dim B'$. Then a linear bijection $\phi : E \rightarrow E'$ between $E \subset V$ and $E' \subset V'$ can be extended to a linear bijection $\phi_V : V \rightarrow V'$ such that $\phi_V(A) = A'$ and $\phi_V(B) = B'$ if and only if $\phi(E \cap A) = E' \cap A'$ and $\phi(E \cap B) = E' \cap B'$.

The lemma is a special case of Lemma 4.3. We refer to the proof there. In the above lemma, every $\mathbf{v} \in V$ can be uniquely decomposed as $\mathbf{v} = \mathbf{v}_A + \mathbf{v}_B$ for $\mathbf{v}_A \in A$ and $\mathbf{v}_B \in B$. Precisely, $\{\mathbf{v}_A\} = (\{\mathbf{v}\} + B) \cap A$ and $\{\mathbf{v}_B\} = (\{\mathbf{v}\} + A) \cap B$. Likewise, every $\mathbf{v}' \in V'$ can be uniquely decomposed as $\mathbf{v}' = \mathbf{v}'_{A'} + \mathbf{v}'_{B'}$ for $\mathbf{v}'_{A'} \in A'$ and $\mathbf{v}'_{B'} \in B'$. Denote $P_{A,B}(E) := [(E + B) \cap A] \oplus [(E + A) \cap B]$. Then $P_{A,B}(E) \supset E$. Define a linear map $\tilde{\phi}_{A,B} : P_{A,B}(E) \rightarrow P_{A',B'}(E')$ by

$$\tilde{\phi}_{A,B}(\mathbf{v}_A) := \phi(\mathbf{v})_{A'}, \quad \tilde{\phi}_{A,B}(\mathbf{v}_B) := \phi(\mathbf{v})_{B'}, \quad \text{for all } \mathbf{v} \in E. \quad (9)$$

From the proof of Lemma 4.3, $\tilde{\phi}_{A,B}$ is a well-defined bijective linear extension of ϕ . Moreover, every linear extension $\phi_V : V \rightarrow V'$ of ϕ that satisfies $\phi_V(A) = A'$ and $\phi_V(B) = B'$ is also an extension of $\tilde{\phi}_{A,B}$.

A *hyperbolic space* is a nonsingular metric space V that has the Witt decomposition $V = V^+ \oplus V^-$, where both V^+ and V^- are maximal totally isotropic subspaces of the same dimension [27]. If $\mathcal{V} := \{V_i\}_{i=0, \dots, k}$ is a self-dual flag of V such that $V_{\lfloor \frac{k}{2} \rfloor} \subset V^+$, then $V_i \subset V^+$ for $i = 0, \dots, \lfloor \frac{k}{2} \rfloor$ and $V^+ \subset V_i$ for $i = \lfloor \frac{k}{2} \rfloor + 1, \dots, k$. There are two cases about \mathcal{V} :

- i) k even: Then $V_{\lfloor \frac{k}{2} \rfloor} = V_{\frac{k}{2}} = V^+$. Denote $\overline{\mathcal{V}} := \mathcal{V}$.
- ii) k odd: Then $\overline{\mathcal{V}} := \{V_0, V_1, \dots, V_{\lfloor \frac{k}{2} \rfloor}, V^+, V_{\lfloor \frac{k}{2} \rfloor + 1}, \dots, V_k\}$ is also a self-dual flag. It is a refinement of \mathcal{V} .

The following theorem is a slight improvement of a main theorem in [11].

THEOREM 4.2. Let $V = V^+ \oplus V^-$ and $V' = V'^+ \oplus V'^-$ be isometric hyperbolic spaces. Let $\mathcal{V} := \{V_i\}_{i=0, \dots, k}$ and $\mathcal{V}' := \{V'_i\}_{i=0, \dots, k}$ be isometric self-dual flags of V and V' respectively such that $V_{\lfloor \frac{k}{2} \rfloor} \subset V^+$ and $V'_{\lfloor \frac{k}{2} \rfloor} \subset V'^+$. Then an isometry $\phi : E \rightarrow E'$ between $E \subset V$ and $E' \subset V'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ that satisfies $\phi_V(V^+) = V'^+$, $\phi_V(V^-) = V'^-$, and $\phi_V(\mathcal{V}) = \mathcal{V}'$, if and only if $\phi(E \cap V^+) = E' \cap V'^+$, $\phi(E \cap V^-) = E' \cap V'^-$, and the induced linear bijection $\tilde{\phi}_{V^+, V^-}$ in (9) is an isometry that satisfies

$$\tilde{\phi}_{V^+, V^-}(P_{V^+, V^-}(E) \cap V_i) = P_{V'^+, V'^-}(E') \cap V'_i \quad \text{for } i = 1, \dots, k.$$

Proof. It suffices to prove the sufficient part. Since ϕ can be extended to $\tilde{\phi}_{V^+, V^-}$ by Lemma 4.1 and the isometry $\tilde{\phi}_{V^+, V^-} : P_{V^+, V^-}(E) \rightarrow P_{V'^+, V'^-}(E')$ satisfies the sufficient conditions about $\phi : E \rightarrow E'$ in the above theorem, without loss of generality, we assume that $\phi = \tilde{\phi}_{V^+, V^-}$ so that

$$\begin{aligned} E &= P_{V^+, V^-}(E) = (E \cap V^+) \oplus (E \cap V^-), \\ E' &= P_{V'^+, V'^-}(E') = (E' \cap V'^+) \oplus (E' \cap V'^-). \end{aligned}$$

Moreover, the sufficient conditions in the theorem still hold if we replace \mathcal{V} by $\overline{\mathcal{V}}$ and \mathcal{V}' by $\overline{\mathcal{V}'}$. Thus we assume $\mathcal{V} = \overline{\mathcal{V}}$ and $\mathcal{V}' = \overline{\mathcal{V}'}$ so that k is even.

By Theorem 3.2, the isometry $\phi : E \rightarrow E'$ can be extended to an isometry $\phi'_V : V \rightarrow V'$ such that $\phi'_V(\mathcal{V}) = \mathcal{V}'$. Then $\phi'_V(V^+) = \phi'_V(V_{\frac{k}{2}}) = V'_{\frac{k}{2}} = V'^+$. Then $\phi_1 := \phi'_V|_{E+V^+}$ is an isometric extension of ϕ sending $E + V^+$ to $E' + V'^+$. Moreover, $\phi_1(V_i) = V'_i$ for $i = 1, \dots, \frac{k}{2}$ (in particular $\phi_1(V^+) = V'^+$). Notice that

$$\phi_1((E + V^+) \cap V^-) = \phi(E \cap V^-) = E' \cap V'^- = (E' + V'^+) \cap V'^-.$$

Applying Theorem 3.2 on $\phi_1 : E + V^+ \rightarrow E' + V'^+$ and the self-dual flags $\{\{\mathbf{0}\} \subset V^- \subset V\}$ and $\{\{\mathbf{0}'\} \subset V'^- \subset V'\}$, the isometry ϕ_1 can be extended to an isometry $\phi_V : V \rightarrow V'$ such that $\phi_V(V^-) = V'^-$. Finally, $\phi_V(V_i) = \phi_1(V_i) = V'_i$ for $i = 1, \dots, \frac{k}{2}$ and

$$\phi_V(V_i) = \phi_V(V_{k-i}^\perp) = \phi_V(V_{k-i})^\perp = V_{k-i}'^\perp = V'_i$$

for $i = \frac{k}{2} + 1, \dots, k$. This shows that $\phi_V(\mathcal{V}) = \mathcal{V}'$. We are done. \square

LEMMA 4.3. Suppose $V = A \oplus B \oplus C$ and $V' = A' \oplus B' \oplus C'$ satisfy that $\dim A = \dim A'$, $\dim B = \dim B'$, and $\dim C = \dim C'$. Then a linear bijection $\phi : E \rightarrow E'$ can be extended to a linear bijection $\phi_V : V \rightarrow V'$ such that $\phi_V(A) = A'$, $\phi_V(B) = B'$, and $\phi_V(C) = C'$, if and only if

$$\phi(E \cap (B + C)) = E' \cap (B' + C'), \quad (10a)$$

$$\phi(E \cap (C + A)) = E' \cap (C' + A'), \quad (10b)$$

$$\phi(E \cap (A + B)) = E' \cap (A' + B'). \quad (10c)$$

Proof. It suffices to prove the sufficient part. Every $\mathbf{v} \in V$ can be uniquely expressed as $\mathbf{v} = \mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C$ for $\mathbf{v}_A \in A$, $\mathbf{v}_B \in B$, $\mathbf{v}_C \in C$. Precisely, $\{\mathbf{v}_A\} = (\{\mathbf{v}\} + B + C) \cap A$ and likewise for \mathbf{v}_B and \mathbf{v}_C . Similarly, every $\mathbf{v}' \in V'$ can be uniquely expressed as $\mathbf{v}' = \mathbf{v}'_{A'} + \mathbf{v}'_{B'} + \mathbf{v}'_{C'}$ for $\mathbf{v}'_{A'} \in A'$, $\mathbf{v}'_{B'} \in B'$, $\mathbf{v}'_{C'} \in C'$. From (10a), we claim that the map $\phi'_A(\mathbf{v}_A) := \phi(\mathbf{v})_{A'}$ for $\mathbf{v} \in E$ is a well-defined linear bijection. To see the well-definedness, suppose $\mathbf{u}, \mathbf{v} \in E$ such that $\mathbf{u}_A = \mathbf{v}_A$, then $\mathbf{u} - \mathbf{v} \in E \cap (B + C)$ and thus $\phi(\mathbf{u}) - \phi(\mathbf{v}) \in E' \cap (B' + C')$ and thus $\phi(\mathbf{u})_{A'} = \phi(\mathbf{v})_{A'}$. Analogous analysis shows that ϕ'_A is a bijection. Then the linearity of ϕ'_A comes from the linearity of ϕ . Likewise, both $\phi'_B(\mathbf{v}_B) := \phi(\mathbf{v})_{B'}$ for $\mathbf{v} \in E$ and $\phi'_C(\mathbf{v}_C) := \phi(\mathbf{v})_{C'}$ for $\mathbf{v} \in E$ are well-defined linear bijections. Extend ϕ'_A (resp. ϕ'_B , ϕ'_C) to a linear bijection $\phi_A : A \rightarrow A'$ (resp. $\phi_B : B \rightarrow B'$, $\phi_C : C \rightarrow C'$). Then $\phi_V := \phi_A \oplus \phi_B \oplus \phi_C$ is a linear bijection from V to V' , such that $\phi_V|_E = \phi$, $\phi_V(A) = A'$, $\phi_V(B) = B'$, and $\phi_V(C) = C'$. \square

In the above proof, denote the sum of the projections of E onto A , B , and C components with respect to $V = A \oplus B \oplus C$ by

$$P_{A,B,C}(E) := [(E + B + C) \cap A] \oplus [(E + C + A) \cap B] \oplus [(E + A + B) \cap C]. \quad (11)$$

When ϕ satisfies the conditions in (10), the map $\tilde{\phi}_{A,B,C} := \phi'_A \oplus \phi'_B \oplus \phi'_C$ is a linear bijection between $P_{A,B,C}(E)$ and $P_{A',B',C'}(E')$. Alternatively, $\tilde{\phi}_{A,B,C}$ may be defined by

$$\tilde{\phi}_{A,B,C}(\mathbf{v}_A) := \phi(\mathbf{v})_{A'}, \quad \tilde{\phi}_{A,B,C}(\mathbf{v}_B) := \phi(\mathbf{v})_{B'}, \quad \tilde{\phi}_{A,B,C}(\mathbf{v}_C) := \phi(\mathbf{v})_{C'}, \quad \text{for } \mathbf{v} \in E. \quad (12)$$

Every extension of ϕ to a linear bijection between V and V' that sends A to A' , B to B' , and C to C' respectively is also an extension of $\tilde{\phi}_{A,B,C}$.

A Witt's decomposition of a generic nonsingular metric space V is

$$V = V^+ \oplus \hat{V} \oplus V^- \quad (13)$$

where V^+ and V^- are maximal totally isotropic spaces of the same dimension and \hat{V} is an anisotropic space orthogonal to both V^+ and V^- . Two nonsingular metric spaces are isometric if and only if their corresponding components in Witt's decompositions are isometric.

THEOREM 4.4. Let $V = V^+ \oplus \widehat{V} \oplus V^-$ and $V' = V'^+ \oplus \widehat{V}' \oplus V'^-$ be Witt's decompositions of isometric spaces V and V' respectively. Let $\mathcal{V} = \{V_i\}_{i=0, \dots, k}$ and $\mathcal{V}' = \{V'_i\}_{i=0, \dots, k}$ be isometric self-dual flags of V and V' respectively such that $V_{\lfloor \frac{k}{2} \rfloor} \subset V^+$ and $V'_{\lfloor \frac{k}{2} \rfloor} \subset V'^+$. Then an isometry $\phi : E \rightarrow E'$ between $E \subset V$ and $E' \subset V'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ such that

$$\phi_V(\widehat{V}) = \widehat{V}', \quad \phi_V(V^+) = V'^+, \quad \phi_V(V^-) = V'^-, \quad \phi_V(\mathcal{V}) = \mathcal{V}', \quad (14)$$

if and only if

$$\phi(E \cap (V^+ + V^-)) = E' \cap (V'^+ + V'^-), \quad (15a)$$

$$\phi(E \cap (V^- + \widehat{V})) = E' \cap (V'^- + \widehat{V}'), \quad (15b)$$

$$\phi(E \cap (\widehat{V} + V^+)) = E' \cap (\widehat{V}' + V'^+), \quad (15c)$$

and the induced map $\widetilde{\phi}_{V^+, \widehat{V}, V^-} : P_{V^+, \widehat{V}, V^-}(E) \rightarrow P_{V'^+, \widehat{V}', V'^-}(E')$ is an isometry that satisfies

$$\widetilde{\phi}_{V^+, \widehat{V}, V^-}(P_{V^+, \widehat{V}, V^-}(E) \cap V_i) = P_{V'^+, \widehat{V}', V'^-}(E') \cap V'_i \quad \text{for } i = 1, \dots, k. \quad (15d)$$

COROLLARY 4.5. An isometry $\phi : E \rightarrow E'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends a given Witt's decomposition $V = V^+ \oplus \widehat{V} \oplus V^-$ to another given Witt's decomposition $V' = V'^+ \oplus \widehat{V}' \oplus V'^-$ if and only if ϕ satisfies (15a), (15b), (15c), and the induced map $\widetilde{\phi}_{V^+, \widehat{V}, V^-}$ is an isometry.

Proof of Theorem 4.4. It suffices to prove the sufficient part. If $\dim \widehat{V} = 0$, then the statement is just Theorem 4.2. Now suppose $\dim \widehat{V} > 0$ so that k is odd. By the same reasons as stated in the proof of Theorem 4.2, without loss of generality, we assume that $\phi = \widetilde{\phi}_{V^+, \widehat{V}, V^-}$ so that

$$\begin{aligned} E &= P_{V^+, \widehat{V}, V^-}(E) = (E \cap V^+) \oplus (E \cap \widehat{V}) \oplus (E \cap V^-), \\ E' &= P_{V'^+, \widehat{V}', V'^-}(E') = (E' \cap V'^+) \oplus (E' \cap \widehat{V}') \oplus (E' \cap V'^-). \end{aligned}$$

Then for $i = \frac{k+1}{2}, \dots, k$, by Lemma 2.1(5)

$$V_i = V_i \cap [\widehat{V} \odot (V^+ \oplus V^-)] = \widehat{V} \odot [V_i \cap (V^+ \oplus V^-)].$$

Likewise for V'_i ($i = \frac{k+1}{2}, \dots, k$). Denote $\widetilde{V}_i := V_i \cap (V^+ \oplus V^-)$ and $\widetilde{V}'_i := V'_i \cap (V'^+ \oplus V'^-)$ for $i = \frac{k+1}{2}, \dots, k$. Then $\widetilde{\mathcal{V}} := \{V_0 \subset \dots \subset V_{\frac{k-1}{2}} \subset \widetilde{V}_{\frac{k+1}{2}} \subset \dots \subset \widetilde{V}_k\}$ is a self-dual flag of $V^+ \oplus V^-$, and $\widetilde{\mathcal{V}}' := \{V'_0 \subset \dots \subset V'_{\frac{k-1}{2}} \subset \widetilde{V}'_{\frac{k+1}{2}} \subset \dots \subset \widetilde{V}'_k\}$ is a self-dual flag of $V'^+ \oplus V'^-$. Notice that

$$\phi = \phi|_{E \cap \widehat{V}} \odot \phi|_{(E \cap V^+) \oplus (E \cap V^-)}.$$

By Witt's theorem, the isometry $\phi|_{E \cap \widehat{V}} : E \cap \widehat{V} \rightarrow E' \cap \widehat{V}'$ can be extended to an isometry $\phi_{\widehat{V}} : \widehat{V} \rightarrow \widehat{V}'$. By applying Theorem 4.2 onto $\phi|_{(E \cap V^+) \oplus (E \cap V^-)}$ and the self-dual flags $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{V}}'$, the isometry $\phi|_{(E \cap V^+) \oplus (E \cap V^-)} : (E \cap V^+) \oplus (E \cap V^-) \rightarrow (E' \cap V'^+) \oplus (E' \cap V'^-)$ can be extended to an isometry $\phi_{V^+ \oplus V^-} : V^+ \oplus V^- \rightarrow V'^+ \oplus V'^-$ such that $\phi_{V^+ \oplus V^-}(V^+) = V'^+$, $\phi_{V^+ \oplus V^-}(V^-) = V'^-$, and $\phi_{V^+ \oplus V^-}(\widetilde{\mathcal{V}}) = \widetilde{\mathcal{V}}'$. Then $\phi_V := \phi_{\widehat{V}} \odot \phi_{V^+ \oplus V^-}$ is a desired isometric linear extension of ϕ that satisfies (14). \square

5. Simultaneous isometry of subspace pairs

Suppose $(V, b) \approx (V', b')$ are nonsingular. Let $E, A \subset V$ and $E', A' \subset V'$ satisfy that $\phi : E \rightarrow E'$ is an isometry and $A \approx A'$. We determine when ϕ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends A to A' .

We begin with the self-dual flags

$$\left\{ \{\mathbf{0}\} \subset A^\perp \cap A \subset A + A^\perp \subset V \right\} \quad \text{and} \quad \left\{ \{\mathbf{0}'\} \subset A'^\perp \cap A' \subset A' + A'^\perp \subset V' \right\}. \quad (16)$$

By Lemma 3.3, the isometry $\phi : E \rightarrow E'$ can be extended to an isometry $\phi'_V : V \rightarrow V'$ such that $A^\perp \cap A \overset{\phi'_V}{\approx} A'^\perp \cap A'$ and $A + A^\perp \overset{\phi'_V}{\approx} A' + A'^\perp$ if and only if

$$\begin{aligned} (C1) \quad & E \cap A^\perp \cap A \overset{\phi}{\approx} E' \cap A'^\perp \cap A', \\ (C2) \quad & E \cap (A + A^\perp) \overset{\phi}{\approx} E' \cap (A' + A'^\perp). \end{aligned}$$

Now if $\phi : E \rightarrow E'$ can be extended to an isometry ϕ_V that sends A to A' , then (C1) and (C2) above and (C3) and (C4) below hold:

$$\begin{aligned} (C3) \quad & E \cap A \overset{\phi}{\approx} E' \cap A', \\ (C4) \quad & E \cap A^\perp \overset{\phi}{\approx} E' \cap A'^\perp. \end{aligned}$$

Obviously, (C3) and (C4) imply (C1).

Suppose the isometry $\phi : E \rightarrow E'$ satisfies (C2)~(C4) (and so (C1)). Then ϕ induces two linear bijections ϕ_A and ϕ_{A^\perp} of metric spaces as follow:

i) The first map is

$$\phi_A : \frac{(E + A^\perp) \cap A}{A^\perp \cap A} \rightarrow \frac{(E' + A'^\perp) \cap A'}{A'^\perp \cap A'}. \quad (17)$$

The motivation of introducing ϕ_A originates from the special case $A^\perp \cap A = \{\mathbf{0}\}$, where $V = A \odot A^\perp$ is nonsingular, and $\frac{(E+A^\perp) \cap A}{A^\perp \cap A} \approx (E + A^\perp) \cap A$ is the projection of E onto A -component with respect to the decomposition $V = A \odot A^\perp$. If ϕ can be extended to an isometry ϕ_V that sends A to A' , then the projection of E onto A -component should be sent isometrically to the projection of E' onto A' -component in $V' = A' \odot A'^\perp$.

Let us define ϕ_A in (17). Given $\mathbf{v} + (A^\perp \cap A) \in \frac{(E+A^\perp) \cap A}{A^\perp \cap A}$, there exist $\mathbf{e} \in E$ and $\mathbf{a}_\perp \in A^\perp$ such that $\mathbf{v} = \mathbf{e} - \mathbf{a}_\perp \in A$, that is, $\mathbf{e} = \mathbf{v} + \mathbf{a}_\perp \in E \cap (A + A^\perp)$. Then $\phi(\mathbf{e}) \in E' \cap (A' + A'^\perp)$ by (C2). So there exist $\mathbf{v}' \in A'$ and $\mathbf{a}'_\perp \in A'^\perp$ such that $\phi(\mathbf{e}) = \mathbf{v}' + \mathbf{a}'_\perp$, that is, $\mathbf{v}' = \phi(\mathbf{e}) - \mathbf{a}'_\perp \in (E' + A'^\perp) \cap A'$. Define

$$\phi_A \left(\mathbf{v} + (A^\perp \cap A) \right) := \mathbf{v}' + (A'^\perp \cap A'). \quad (18)$$

The space $\frac{A+A^\perp}{A^\perp \cap A}$ (and so $\frac{(E+A^\perp) \cap A}{A^\perp \cap A}$) carries a metric \bar{b} induced from b :

$$\bar{b} \left(\mathbf{v}_1 + (A^\perp \cap A), \mathbf{v}_2 + (A^\perp \cap A) \right) := b(\mathbf{v}_1, \mathbf{v}_2) \quad \text{for } \mathbf{v}_1, \mathbf{v}_2 \in A + A^\perp. \quad (19)$$

Analogously, $\frac{A'+A'^\perp}{A'^\perp \cap A'}$ (and so $\frac{(E'+A'^\perp) \cap A'}{A'^\perp \cap A'}$) carries a metric \bar{b}' induced from b' .

LEMMA 5.1. ϕ_A is a well-defined linear bijection of metric spaces.

Proof. To establish the well-definedness of ϕ_A , suppose $\mathbf{v}_1 + (A^\perp \cap A) = \mathbf{v}_2 + (A^\perp \cap A)$ in $\frac{(E+A^\perp) \cap A}{A^\perp \cap A}$, where $\mathbf{v}_i = \mathbf{e}_i - \mathbf{a}_{i\perp} \in A$ for $\mathbf{e}_i \in E$ and $\mathbf{a}_{i\perp} \in A^\perp$, $i = 1, 2$. Then $\mathbf{v}_1 - \mathbf{v}_2 \in A^\perp \cap A$ and so

$$(\mathbf{e}_1 - \mathbf{e}_2) - (\mathbf{a}_{1\perp} - \mathbf{a}_{2\perp}) \in A^\perp \cap A.$$

Thus $\mathbf{e}_1 - \mathbf{e}_2 \in E \cap A^\perp$ by $\mathbf{a}_{1\perp} - \mathbf{a}_{2\perp} \in A^\perp$, and $\phi(\mathbf{e}_1) - \phi(\mathbf{e}_2) \in E' \cap A'^\perp$ by (C4). Notice that $\phi(\mathbf{e}_i) \in E' \cap (A' + A'^\perp)$ by the preceding discussion in defining ϕ_A . Suppose $\phi(\mathbf{e}_i) = \mathbf{v}'_i + \mathbf{a}'_{i\perp}$ for $\mathbf{v}'_i \in A'$ and $\mathbf{a}'_{i\perp} \in A'^\perp$, $i = 1, 2$. Then

$$\mathbf{v}'_1 - \mathbf{v}'_2 = (\phi(\mathbf{e}_1) - \phi(\mathbf{e}_2)) - (\mathbf{a}'_{1\perp} - \mathbf{a}'_{2\perp}). \quad (20)$$

Both $\phi(\mathbf{e}_1) - \phi(\mathbf{e}_2)$ and $\mathbf{a}'_{1\perp} - \mathbf{a}'_{2\perp}$ are in A'^\perp . So $\mathbf{v}'_1 - \mathbf{v}'_2 \in A'^\perp \cap A'$; that is,

$$\mathbf{v}'_1 + (A'^\perp \cap A') = \mathbf{v}'_2 + (A'^\perp \cap A').$$

Hence ϕ_A is well-defined.

The map ϕ_A is linear by its well-definedness and the linearity of ϕ , and ϕ_A is a bijection since ϕ_A^{-1} can be defined by ϕ^{-1} in a similar fashion. \square

ii) Likewise, we define the second map

$$\phi_{A^\perp} : \frac{(E + A) \cap A^\perp}{A^\perp \cap A} \rightarrow \frac{(E' + A') \cap A'^\perp}{A'^\perp \cap A'}.$$
 (21)

Given $\mathbf{v} + (A^\perp \cap A) \in \frac{(E+A) \cap A^\perp}{A^\perp \cap A}$, there are $\mathbf{e} \in E$ and $\mathbf{a} \in A$ such that $\mathbf{v} = \mathbf{e} - \mathbf{a} \in A^\perp$. So $\mathbf{e} = \mathbf{a} + \mathbf{v} \in E \cap (A + A^\perp)$, and thus $\phi(\mathbf{e}) \in E' \cap (A' + A'^\perp)$ by (C2). There are $\mathbf{a}' \in A'$ and $\mathbf{v}' \in A'^\perp$ such that $\phi(\mathbf{e}) = \mathbf{a}' + \mathbf{v}'$. Define

$$\phi_{A^\perp}(\mathbf{v} + (A^\perp \cap A)) := \mathbf{v}' + (A'^\perp \cap A').$$
 (22)

LEMMA 5.2. ϕ_{A^\perp} is a well-defined linear bijection of metric spaces.

The proof is analogous and so skipped.

The following is one of the main theorems of this paper.

THEOREM 5.3. *Let (V, b) and (V', b') be isometric nonsingular symmetric metric spaces over \mathbb{F} . Let subspaces $E, A \subset V$ and $E', A' \subset V'$ satisfy that $A \approx A'$ and that $\phi : E \rightarrow E'$ is an isometry. Then ϕ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends A to A' if and only if (C2)~(C4) hold and one of ϕ_A and ϕ_{A^\perp} is an isometry.*

Since ϕ is an isometry, one of ϕ_A and ϕ_{A^\perp} is an isometry if and only if both of them are isometries by the constructions of ϕ_A and ϕ_{A^\perp} .

REMARK 5.4. Theorem 5.3 embraces several other results. If $E \subset A$, then ϕ_{A^\perp} is trivial, and Theorem 5.3 is a combination of Lemma 2.3 and Witt's theorem. If $E \perp A$, then ϕ_A is trivial, and Theorem 5.3 is Corollary 2.5. If A (resp. A^\perp) is totally isotropic, then ϕ_A (resp. ϕ_{A^\perp}) is trivial, and Theorem 5.3 is equivalent to Lemma 3.3. If $E \cap (A + A^\perp) = (E \cap A) + (E \cap A^\perp)$, then Theorem 5.3 becomes Corollary 5.5, which plays an important role in determining the isometry of two generic flags in Section 6.

Proof of Theorem 5.3. It suffices to prove the sufficient part. Recall that (C1) is implied by (C3) and (C4).

i) We shall extend $\phi|_{E \cap (A + A^\perp)}$ to an isometry $\widehat{\phi} : A + A^\perp \rightarrow A' + A'^\perp$ that sends A to A' , and

$$b(\mathbf{e}, \mathbf{v}) = b'(\phi(\mathbf{e}), \widehat{\phi}(\mathbf{v}))$$
 (23)

for all $\mathbf{e} \in E$ and $\mathbf{v} \in [E \cap (A + A^\perp)] + (A^\perp \cap A)$.

Step 1: Decompose $E \cap (A + A^\perp)$.

Select E_A and E_{A^\perp} such that

$$E \cap A = (E \cap A^\perp \cap A) \odot E_A,$$
 (24)

$$E \cap A^\perp = (E \cap A^\perp \cap A) \odot E_{A^\perp}.$$
 (25)

Then $E_A \cap (E \cap A^\perp) = (E_A \cap A) \cap (E \cap A^\perp) = E_A \cap (E \cap A^\perp \cap A) = \{\mathbf{0}\}$. So

$$(E \cap A) + (E \cap A^\perp) = (E \cap A^\perp \cap A) \odot E_A \odot E_{A^\perp}.$$
 (26)

Select E_{A,A^\perp} such that

$$E \cap (A + A^\perp) = [(E \cap A) + (E \cap A^\perp)] \oplus E_{A,A^\perp} \quad (27)$$

$$= (E \cap A^\perp \cap A) \odot [(E_A \odot E_{A^\perp}) \oplus E_{A,A^\perp}]. \quad (28)$$

Step 2: Decompose $A + A^\perp$ in accordance with (28).

By the selection of E_{A,A^\perp} ,

$$E_{A,A^\perp} \cap A = \{\mathbf{0}\}, \quad E_{A,A^\perp} \cap A^\perp = \{\mathbf{0}\}, \quad E_{A,A^\perp} \subset A + A^\perp.$$

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis of E_{A,A^\perp} . Then $\mathbf{v}_i = \mathbf{a}_i + \mathbf{a}_{i\perp}$ for some $\mathbf{a}_i \in A$ and $\mathbf{a}_{i\perp} \in A^\perp$, $i = 1, \dots, k$. The following observations are useful:

- (a) The choices of \mathbf{a}_i and $\mathbf{a}_{i\perp}$ for a given \mathbf{v}_i are not unique if $A^\perp \cap A \neq \{\mathbf{0}\}$.
- (b) The set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is always linearly independent. Suppose $\sum_{i=1}^k t_i \mathbf{a}_i = \mathbf{0}$ for $t_i \in \mathbb{F}$, then $\sum_{i=1}^k t_i \mathbf{v}_i = \sum_{i=1}^k t_i \mathbf{a}_{i\perp} \in E_{A,A^\perp} \cap A^\perp = \{\mathbf{0}\}$. This forces $t_1 = \dots = t_k = 0$ by the linear independence of $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.
- (c) Likewise, $\{\mathbf{a}_{1\perp}, \dots, \mathbf{a}_{k\perp}\}$ is linearly independent.
- (d) Denote

$$E_{(A)} := \text{span}\{\mathbf{a}_1, \dots, \mathbf{a}_k\}, \quad E_{(A^\perp)} := \text{span}\{\mathbf{a}_{1\perp}, \dots, \mathbf{a}_{k\perp}\}. \quad (29)$$

Then $E_{(A)} \subset A$, $E_{(A^\perp)} \subset A^\perp$, and

$$E_{A,A^\perp} \subset E_{(A)} \odot E_{(A^\perp)}. \quad (30)$$

We claim that

$$(E_{(A)} \odot E_{(A^\perp)}) \cap [(E \cap A) + (E \cap A^\perp) + (A^\perp \cap A)] = \{\mathbf{0}\}. \quad (31)$$

Suppose $\mathbf{x} = \mathbf{a} + \mathbf{b}_\perp = \mathbf{e}_A + \mathbf{e}_{A^\perp} + \mathbf{a}_0$ for $\mathbf{a} \in E_{(A)}$, $\mathbf{b}_\perp \in E_{(A^\perp)}$, $\mathbf{e}_A \in E \cap A$, $\mathbf{e}_{A^\perp} \in E \cap A^\perp$, and $\mathbf{a}_0 \in A^\perp \cap A$, then by (29), there exist $\mathbf{a}_\perp \in E_{(A^\perp)}$ and $\mathbf{e} \in E_{A,A^\perp}$ such that $\mathbf{e} = \mathbf{a} + \mathbf{a}_\perp$. So

$$\mathbf{e} = \mathbf{a} + \mathbf{b}_\perp + \mathbf{a}_\perp - \mathbf{b}_\perp = \mathbf{e}_A + \mathbf{e}_{A^\perp} + (\mathbf{a}_0 + \mathbf{a}_\perp - \mathbf{b}_\perp). \quad (32)$$

Then $\mathbf{a}_0 + \mathbf{a}_\perp - \mathbf{b}_\perp \in A^\perp$ by $\mathbf{a}_0 \in A^\perp \cap A$ and $\mathbf{a}_\perp, \mathbf{b}_\perp \in E_{(A^\perp)} \subset A^\perp$. Moreover, $\mathbf{a}_0 + \mathbf{a}_\perp - \mathbf{b}_\perp \in E$ by (32) and the fact $\mathbf{e}, \mathbf{e}_A, \mathbf{e}_{A^\perp} \in E$. So $\mathbf{a}_0 + \mathbf{a}_\perp - \mathbf{b}_\perp \in E \cap A^\perp$. Thus $\mathbf{e} \in (E \cap A) + (E \cap A^\perp)$ by (32). This forces $\mathbf{e} = \mathbf{0}$ by $\mathbf{e} \in E_{A,A^\perp}$ and (27). So $\mathbf{a} = \mathbf{0}$. Likewise, $\mathbf{b}_\perp = \mathbf{0}$. Therefore, $\mathbf{x} = \mathbf{0}$ and (31) hold.

- (e) By (24) and (25),

$$(E \cap A) + (A^\perp \cap A) = (A^\perp \cap A) \odot E_A, \quad (33)$$

$$(E \cap A^\perp) + (A^\perp \cap A) = (A^\perp \cap A) \odot E_{A^\perp}. \quad (34)$$

We claim that

$$(E + A^\perp) \cap A = ([E \cap (A + A^\perp)] + A^\perp) \cap A. \quad (35)$$

Every $\mathbf{x} \in (E + A^\perp) \cap A$ can be written as $\mathbf{x} = \mathbf{e} - \mathbf{a}_\perp = \mathbf{a}$ for $\mathbf{e} \in E$, $\mathbf{a}_\perp \in A^\perp$, and $\mathbf{a} \in A$. Then $\mathbf{e} = \mathbf{a} + \mathbf{a}_\perp \in E \cap (A + A^\perp)$ and so $\mathbf{x} \in ([E \cap (A + A^\perp)] + A^\perp) \cap A$. This proves (35). By (35), (28), and Lemma 2.1(5),

$$\begin{aligned} (E + A^\perp) \cap A &= [(E \cap A^\perp \cap A) + E_A + E_{A^\perp} + E_{A,A^\perp} + A^\perp] \cap A \\ &= (E_A + E_{A,A^\perp} + A^\perp) \cap A \\ &= (E_A + E_{(A)} + A^\perp) \cap A \\ &= (A^\perp \cap A) \odot (E_A \oplus E_{(A)}). \end{aligned} \quad (36)$$

Likewise,

$$(E + A) \cap A^\perp = (A^\perp \cap A) \odot (E_{A^\perp} \oplus E_{(A^\perp)}). \quad (37)$$

By (31), (36) and (37),

$$(E + A^\perp) \cap A + (E + A) \cap A^\perp = (A^\perp \cap A) \odot (E_A \oplus E_{(A)}) \odot (E_{A^\perp} \oplus E_{(A^\perp)}). \quad (38)$$

By (36), $E_A \oplus E_{(A)}$ can be extended to V_A such that

$$A = (A^\perp \cap A) \odot V_A. \quad (39)$$

By (37), $E_{A^\perp} \oplus E_{(A^\perp)}$ can be extended to V_{A^\perp} such that

$$A^\perp = (A^\perp \cap A) \odot V_{A^\perp}. \quad (40)$$

Then

$$A + A^\perp = (A^\perp \cap A) \odot V_A \odot V_{A^\perp}. \quad (41)$$

The decompositions (28) and (41) satisfy that

$$E \cap A^\perp \cap A \subset A^\perp \cap A, \quad E_A \subset V_A, \quad E_{A^\perp} \subset V_{A^\perp}, \quad E_{A,A^\perp} \subset V_A \odot V_{A^\perp}. \quad (42)$$

Both V_A and V_{A^\perp} are nonsingular metric subspaces by (41).

Step 3: Recall that $\frac{A+A^\perp}{A^\perp \cap A}$ has a metric \bar{b} defined by (19). The canonical quotient map $\rho : A + A^\perp \rightarrow \frac{A+A^\perp}{A^\perp \cap A}$ is metric-preserving. By (33), (36), (39), (41), ρ acts as isometries in the following commutative diagram with inclusion maps:

$$\begin{array}{ccccccc} E_A & \hookrightarrow & E_A \oplus E_{(A)} & \hookrightarrow & V_A & \hookrightarrow & V_A \odot V_{A^\perp} \\ \rho \downarrow & & \rho \downarrow & & \rho \downarrow & & \rho \downarrow \\ \frac{(E \cap A) + (A^\perp \cap A)}{A^\perp \cap A} & \hookrightarrow & \frac{(E + A^\perp) \cap A}{A^\perp \cap A} & \hookrightarrow & \frac{A}{A^\perp \cap A} & \hookrightarrow & \frac{A + A^\perp}{A^\perp \cap A} \end{array} \quad (43)$$

Similarly, ρ acts as isometries on the commutative diagram obtained by interchanging A and A^\perp in (43).

Step 4: In V' , denote $E'_{A'} := \phi(E_A)$, $E'_{A'^\perp} := \phi(E_{A^\perp})$, $E'_{A',A'^\perp} := \phi(E_{A,A^\perp})$. Then by (C1), (C3), and (C4),

$$E' \cap A' = (E' \cap A'^\perp \cap A') \odot E'_{A'}, \quad (44)$$

$$E' \cap A'^\perp = (E' \cap A'^\perp \cap A') \odot E'_{A'^\perp}. \quad (45)$$

By (C1) and (C4),

$$E' \cap (A' + A'^\perp) = (E' \cap A'^\perp \cap A') \odot [(E'_{A'} \odot E'_{A'^\perp}) \oplus E'_{A',A'^\perp}]. \quad (46)$$

The subspace E'_{A',A'^\perp} has a basis $\{\mathbf{v}'_1, \dots, \mathbf{v}'_k\}$ where $\mathbf{v}'_i := \phi(\mathbf{v}_i)$ for $i = 1, \dots, k$. Write $\mathbf{v}'_i = \mathbf{a}'_i + \mathbf{a}'_{i\perp}$ for certain $\mathbf{a}'_i \in A'$ and $\mathbf{a}'_{i\perp} \in A'^\perp$. Denote

$$E'_{(A')} := \text{span}\{\mathbf{a}'_1, \dots, \mathbf{a}'_k\}, \quad E'_{(A'^\perp)} := \text{span}\{\mathbf{a}'_{1\perp}, \dots, \mathbf{a}'_{k\perp}\}. \quad (47)$$

Extend $E'_{A'} \oplus E'_{(A')}$ to $V'_{A'}$ such that

$$A' = (A'^\perp \cap A') \odot V'_{A'}. \quad (48)$$

Extend $E'_{A'^\perp} \oplus E'_{(A'^\perp)}$ to $V'_{A'^\perp}$ such that

$$A'^\perp = (A'^\perp \cap A') \odot V'_{A'^\perp}. \quad (49)$$

Then

$$A' + A'^\perp = (A'^\perp \cap A') \odot V'_{A'} \odot V'_{A'^\perp}. \quad (50)$$

The canonical quotient map $\rho' : A' + A'^\perp \rightarrow \frac{A' + A'^\perp}{A'^\perp \cap A'}$ is metric-preserving, and ρ' acts as isometries in the commutative diagram:

$$\begin{array}{ccccccc}
 E'_{A'} & \hookrightarrow & E'_{A'} \oplus E'_{(A')} & \hookrightarrow & V'_{A'} & \hookrightarrow & V'_{A'} \odot V'_{A'^\perp} \\
 \rho' \downarrow & & \rho' \downarrow & & \rho' \downarrow & & \rho' \downarrow \\
 \frac{(E' \cap A') + (A'^\perp \cap A')}{A'^\perp \cap A'} & \hookrightarrow & \frac{(E' + A'^\perp) \cap A'}{A'^\perp \cap A'} & \hookrightarrow & \frac{A'}{A'^\perp \cap A'} & \hookrightarrow & \frac{A' + A'^\perp}{A'^\perp \cap A'}
 \end{array} \quad (51)$$

Likewise, ρ' acts as isometries on the commutative diagram obtained by interchanging A' and A'^\perp in (51).

Step 5: We now extend $\phi|_{E \cap (A + A^\perp)} : E \cap (A + A^\perp) \rightarrow E' \cap (A' + A'^\perp)$ to an isometry

$\widehat{\phi} : A + A^\perp \rightarrow A' + A'^\perp$, such that $A \approx A'$, and $b(\mathbf{e}, \mathbf{v}) = b'(\phi(\mathbf{e}), \widehat{\phi}(\mathbf{v}))$ for all $\mathbf{e} \in E$ and $\mathbf{v} \in [E \cap (A + A^\perp)] + (A^\perp \cap A)$.

The isometry $\rho'|_{E'_{A'} \oplus E'_{(A')}} : E'_{A'} \oplus E'_{(A')} \rightarrow \frac{(E' + A'^\perp) \cap A'}{A'^\perp \cap A'}$ is invertible. By the sufficient conditions in the theorem, ϕ_A defined in (17) is an isometry. So

$$(\rho'|_{E'_{A'} \oplus E'_{(A')}})^{-1} \circ \phi_A \circ (\rho|_{E_A \oplus E_{(A)}}) : E_A \oplus E_{(A)} \rightarrow E'_{A'} \oplus E'_{(A')} \quad (52)$$

is an isometry. Moreover, by the construction of ϕ_A , the map (52) restricted on E_A is equal to $\phi|_{E_A}$, and it sends \mathbf{a}_i to \mathbf{a}'_i for $i = 1, \dots, k$. Apply Witt's theorem on $E_A \oplus E_{(A)} \subset V_A$ and $E'_{A'} \oplus E'_{(A')} \subset V'_{A'}$ where $V_A \approx \frac{A}{A^\perp \cap A} \approx \frac{A'}{A'^\perp \cap A'} \approx V'_{A'}$ are nonsingular. The isometry (52) can be extended to an isometry

$$\phi_1 : V_A \rightarrow V'_{A'}. \quad (53)$$

Likewise,

$$(\rho'|_{E'_{A'^\perp} \oplus E'_{(A'^\perp)}})^{-1} \circ \phi_{A^\perp} \circ (\rho|_{E_{A^\perp} \oplus E_{(A^\perp)}}) : E_{A^\perp} \oplus E_{(A^\perp)} \rightarrow E'_{A'^\perp} \oplus E'_{(A'^\perp)} \quad (54)$$

is an isometry. Its restriction on E_{A^\perp} is equal to $\phi|_{E_{A^\perp}}$, and it sends $\mathbf{a}_{i\perp}$ to $\mathbf{a}'_{i\perp}$ for $i = 1, \dots, k$. By Witt's theorem, the map (54) can be extended to an isometry

$$\phi_2 : V_{A^\perp} \rightarrow V'_{A'^\perp}. \quad (55)$$

Next, we extend $\phi|_{E \cap A^\perp \cap A} : E \cap A^\perp \cap A \rightarrow E' \cap A'^\perp \cap A'$ to an isometry

$$\phi_0 : A^\perp \cap A \rightarrow A'^\perp \cap A'$$

such that $b(\mathbf{e}, \mathbf{v}) = b'(\phi(\mathbf{e}), \phi_0(\mathbf{v}))$ for all $\mathbf{e} \in E$ and $\mathbf{v} \in A^\perp \cap A$.

(a) Select \widetilde{E} such that

$$E = [E \cap (A + A^\perp)] \oplus \widetilde{E}. \quad (56)$$

Denote $\widetilde{E}' := \phi(\widetilde{E})$. Then (C2) implies that

$$E' = [E' \cap (A' + A'^\perp)] \oplus \widetilde{E}'. \quad (57)$$

Let $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ be a basis of $E^\perp \cap E \cap A^\perp \cap A$. On one hand, extend $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ to a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ of $E \cap A^\perp \cap A$. On the other hand, extend $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ to a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell, \mathbf{w}_{p+1}, \dots, \mathbf{w}_q\}$ of $E^\perp \cap A^\perp \cap A$. Then $\{\mathbf{w}_1, \dots, \mathbf{w}_q\}$ is a basis of $(E \cap A^\perp \cap A) + (E^\perp \cap A^\perp \cap A)$. Extend $\{\mathbf{w}_1, \dots, \mathbf{w}_q\}$ to a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of $A^\perp \cap A$. So $\ell \leq p \leq q \leq m$.

(b) Since V' is nonsingular, and $(A'^\perp \cap A') + \widetilde{E}'^\perp = V'$ by taking orthogonal complement on $(A' + A'^\perp) \cap \widetilde{E}' = \{\mathbf{0}\}$, the Riesz representation $\tau' : A'^\perp \cap A' \rightarrow (\widetilde{E}')^*$, defined by

$$\tau'(\mathbf{v}')(\mathbf{e}') := b'(\mathbf{e}', \mathbf{v}') \quad \text{for } \mathbf{v}' \in A'^\perp \cap A', \mathbf{e}' \in \widetilde{E}', \quad (58)$$

is surjective. Given one

$$\mathbf{w}_i \in (A^\perp \cap A) - \left[(E \cap A^\perp \cap A) + (E^\perp \cap A^\perp \cap A) \right] \quad \text{for } i = q+1, \dots, m,$$

we define $\varphi_{\mathbf{w}_i} : \tilde{E}' \rightarrow \mathbb{F}$ by

$$\varphi_{\mathbf{w}_i}(\mathbf{e}') := b(\phi^{-1}(\mathbf{e}'), \mathbf{w}_i) \quad \text{for } \mathbf{e}' \in \tilde{E}'. \quad (59)$$

Then $\varphi_{\mathbf{w}_i} \in (\tilde{E}')^*$. There exists $\mathbf{w}'_i \in A'^\perp \cap A'$ such that $\tau'(\mathbf{w}'_i) = \varphi_{\mathbf{w}_i}$, that is,

$$b'(\mathbf{e}', \mathbf{w}'_i) = b(\phi^{-1}(\mathbf{e}'), \mathbf{w}_i) \quad \text{for } \mathbf{e}' \in \tilde{E}'.$$

Equivalently, $b'(\phi(\mathbf{e}), \mathbf{w}'_i) = b(\mathbf{e}, \mathbf{w}_i)$ for $\mathbf{e} \in \tilde{E}$ and $i = q+1, \dots, m$.

(c) By (C1) and Lemma 2.2(2), $\{\phi(\mathbf{w}_1), \dots, \phi(\mathbf{w}_\ell)\}$ is a basis of $E'^\perp \cap E' \cap A'^\perp \cap A'$. Then

$$\begin{aligned} \dim(E^\perp \cap A^\perp \cap A) &= \dim V - \dim(E^\perp \cap A^\perp \cap A)^\perp \\ &= \dim V - \dim(E + A + A^\perp) \\ &= \dim V - \dim E - \dim(A + A^\perp) + \dim[E \cap (A + A^\perp)]. \end{aligned}$$

Notice that $A + A^\perp \approx A' + A'^\perp$ by Lemma 2.2. So $\dim(E^\perp \cap A^\perp \cap A) = \dim(E'^\perp \cap A'^\perp \cap A')$. Therefore, $\{\phi(\mathbf{w}_1), \dots, \phi(\mathbf{w}_\ell)\}$ can be extended to a basis $\{\phi(\mathbf{w}_1), \dots, \phi(\mathbf{w}_\ell), \mathbf{w}'_{p+1}, \dots, \mathbf{w}'_q\}$ of $E'^\perp \cap A'^\perp \cap A'$.

(d) Define the linear map $\phi_0 : A^\perp \cap A \rightarrow A'^\perp \cap A'$ by

$$\phi_0(\mathbf{w}_i) := \begin{cases} \phi(\mathbf{w}_i) & \text{for } i = 1, \dots, \ell, \ell+1, \dots, p \\ \mathbf{w}'_i & \text{for } i = p+1, \dots, q, q+1, \dots, m \end{cases} \quad (60)$$

Then (60) leads to the following conclusions:

- (1) The map ϕ_0 is an extension of $\phi|_{E \cap A^\perp \cap A}$.
- (2) By (c), the restricted map $\phi_0|_{E^\perp \cap A^\perp \cap A}$ is a bijection.
- (3) $b'(\phi(\mathbf{e}), \phi_0(\mathbf{w}_i)) = b'(\phi(\mathbf{e}), \phi(\mathbf{w}_i)) = b(\mathbf{e}, \mathbf{w}_i)$ for $\mathbf{e} \in \tilde{E}$ and $i = 1, \dots, p$.
- (4) By (c), $b'(\phi(\mathbf{e}), \phi_0(\mathbf{w}_i)) = b'(\phi(\mathbf{e}), \mathbf{w}'_i) = 0 = b(\mathbf{e}, \mathbf{w}_i)$ for $\mathbf{e} \in \tilde{E}$ and $i = p+1, \dots, q$.
- (5) By (b), $b'(\phi(\mathbf{e}), \phi_0(\mathbf{w}_i)) = b'(\phi(\mathbf{e}), \mathbf{w}'_i) = b(\mathbf{e}, \mathbf{w}_i)$ for $\mathbf{e} \in \tilde{E}$ and $i = q+1, \dots, m$.
- (6) By Lemma 2.1(3), $b'(\phi(\mathbf{e}), \phi_0(\mathbf{v})) = 0 = b(\mathbf{e}, \mathbf{v})$ for $\mathbf{e} \in E \cap (A + A^\perp)$ and $\mathbf{v} \in A^\perp \cap A$. Hence $b'(\phi(\mathbf{e}), \phi_0(\mathbf{v})) = b(\mathbf{e}, \mathbf{v})$ for $\mathbf{e} \in E$ and $\mathbf{v} \in A^\perp \cap A$. We claim that ϕ_0 is an injection, so that it is an isometry between totally isotropic spaces $A^\perp \cap A$ and $A'^\perp \cap A'$. If $\phi_0(\mathbf{v}) = \mathbf{0}'$ for $\mathbf{v} \in A^\perp \cap A$, then $b(\mathbf{e}, \mathbf{v}) = b'(\phi(\mathbf{e}), \phi_0(\mathbf{v})) = 0$ for all $\mathbf{e} \in E$. Then $\mathbf{v} \in E^\perp \cap A^\perp \cap A$. Then $\mathbf{v} = \mathbf{0}$ since $\phi_0|_{E^\perp \cap A^\perp \cap A}$ is a bijection by (2). So ϕ_0 is an injection. Therefore, ϕ_0 in (60) is an isometry that extends $\phi|_{E \cap A^\perp \cap A}$, and $b'(\phi(\mathbf{e}), \phi_0(\mathbf{v})) = b(\mathbf{e}, \mathbf{v})$ for $\mathbf{e} \in E$ and $\mathbf{v} \in A^\perp \cap A$.

By (41) and (50), the map

$$\hat{\phi} := \phi_0 \odot \phi_1 \odot \phi_2 \quad (61)$$

is an isometry from $A + A^\perp$ to $A' + A'^\perp$, where

$$\hat{\phi}(A) = \phi_0(A^\perp \cap A) \odot \phi_1(V_A) = (A'^\perp \cap A') \odot V_{A'} = A'.$$

By (28), (46), and the above discussion, $\hat{\phi}$ is an extension of $\phi|_{E \cap (A + A^\perp)}$. Clearly $b'(\phi(\mathbf{e}), \hat{\phi}(\mathbf{v})) = b(\mathbf{e}, \mathbf{v})$ for $\mathbf{e} \in E$ and $\mathbf{v} \in [E \cap (A + A^\perp)] + (A^\perp \cap A)$. This is (23).

- ii) Now the isometries $\hat{\phi} : A + A^\perp \rightarrow A' + A'^\perp$ and $\phi : E \rightarrow E'$ agree on $\text{Dom}(\hat{\phi}) \cap \text{Dom}(\phi) = E \cap (A + A^\perp)$. However, the combination of these two isometries may not be an isometry — given $\tilde{\mathbf{e}} \in \tilde{E}$ and $\mathbf{v} \in (A + A^\perp) - ([E \cap (A + A^\perp)] + (A^\perp \cap A))$, it is not guaranteed that $b(\tilde{\mathbf{e}}, \mathbf{v}) = b'(\phi(\tilde{\mathbf{e}}), \hat{\phi}(\mathbf{v}))$. So $\hat{\phi}$ should be adjusted slightly.

Choose a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ of $[E \cap (A + A^\perp)] + (A^\perp \cap A)$. Then extend it to a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ of $A + A^\perp$. Given one \mathbf{u}_i for $i = r + 1, \dots, s$, we define $\mu_{\mathbf{u}_i} : \tilde{E}' \rightarrow \mathbb{F}$ by

$$\mu_{\mathbf{u}_i}(\mathbf{e}') := b'(\mathbf{e}', \widehat{\phi}(\mathbf{u}_i)) - b(\phi^{-1}(\mathbf{e}'), \mathbf{u}_i) \quad \text{for } \mathbf{e}' \in \tilde{E}'.$$

Then $\mu_{\mathbf{u}_i} \in (\tilde{E}')^*$. By the surjectivity of τ' defined in (58), we can select $\mathbf{v}'_{\mathbf{u}_i} \in A'^\perp \cap A'$ such that

$$b'(\mathbf{e}', \widehat{\phi}(\mathbf{u}_i)) - b(\phi^{-1}(\mathbf{e}'), \mathbf{u}_i) = b'(\mathbf{e}', \mathbf{v}'_{\mathbf{u}_i}) \quad \text{for } \mathbf{e}' \in \tilde{E}'.$$

Let $\mathbf{e}' := \phi(\mathbf{e})$ for $\mathbf{e} \in \tilde{E}$. Then

$$b'(\phi(\mathbf{e}), \widehat{\phi}(\mathbf{u}_i) - \mathbf{v}'_{\mathbf{u}_i}) = b(\mathbf{e}, \mathbf{u}_i) \quad \text{for } \mathbf{e} \in \tilde{E}. \quad (62)$$

Define the linear map $\tilde{\phi} : E + A + A^\perp \rightarrow E' + A' + A'^\perp$ by

$$\begin{cases} \tilde{\phi}(\mathbf{u}_i) := \widehat{\phi}(\mathbf{u}_i), & \text{for } i = 1, \dots, r \\ \tilde{\phi}(\mathbf{u}_i) := \widehat{\phi}(\mathbf{u}_i) - \mathbf{v}'_{\mathbf{u}_i}, & \text{for } i = r + 1, \dots, s \\ \tilde{\phi}(\mathbf{e}) := \phi(\mathbf{e}), & \text{for } \mathbf{e} \in \tilde{E} \end{cases} \quad (63)$$

The construction (63) leads to the following conclusions:

- (1) $\tilde{\phi}$ is an extension of ϕ since $\tilde{\phi}|_E = \widehat{\phi}|_{E \cap (A + A^\perp)} \oplus \phi|_{\tilde{E}} = \phi$.
- (2) $\tilde{\phi}(A) \subset \widehat{\phi}(A) + (A'^\perp \cap A') = A'$.
- (3) $\tilde{\phi}$ is metric-preserving.
- (4) We claim that $\tilde{\phi}$ is a bijection. By (56) and (57),

$$E + A + A^\perp = (A + A^\perp) \oplus \tilde{E}, \quad E' + A' + A'^\perp = (A' + A'^\perp) \oplus \tilde{E}', \quad (64)$$

Choose a basis $\{\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_t\}$ of \tilde{E} . Then $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_t\}$ is a basis of $E + A + A^\perp$, and $\{\widehat{\phi}(\mathbf{u}_1), \dots, \widehat{\phi}(\mathbf{u}_s), \phi(\tilde{\mathbf{e}}_1), \dots, \phi(\tilde{\mathbf{e}}_t)\}$ is a basis of $E' + A' + A'^\perp$. By (63), the matrix representation of $\tilde{\phi}$ with respect to the above two bases is a unit upper triangular matrix, which is nonsingular. So the map $\tilde{\phi}$ is a bijection.

Therefore, $\tilde{\phi}$ defined in (63) is an isometry that extends ϕ , and $\tilde{\phi}(A) = A'$. By Witt's theorem, $\tilde{\phi}$ can be extended to an isometry $\phi_V : V \rightarrow V'$. Clearly the isometry ϕ_V is an extension of $\phi : E \rightarrow E'$ that sends A to A' . We are done. \square

By (24), (25), (28), (36), and (37), the following three equalities about $E, A \subset V$ are equivalent:

$$(E + A^\perp) \cap A = E \cap A, \quad (65a)$$

$$(E + A) \cap A^\perp = E \cap A^\perp, \quad (65b)$$

$$E \cap (A + A^\perp) = (E \cap A) + (E \cap A^\perp). \quad (65c)$$

Theorem 5.3 implies the following result:

COROLLARY 5.5. Let $E, A \subset V$ and $E', A' \subset V'$ satisfy that $E \approx E'$, $A \approx A'$, $(E + A^\perp) \cap A = E \cap A$ and $(E' + A'^\perp) \cap A' = E' \cap A'$. Then an isometry $\phi : E \rightarrow E'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ that also sends A to A' if and only if (C3) and (C4) hold.

Proof. It suffices to prove the sufficient part. Suppose that (C3): $E \cap A \stackrel{\phi}{\approx} E' \cap A'$ and (C4): $E \cap A^\perp \stackrel{\phi}{\approx} E' \cap A'^\perp$ hold. By the sufficient conditions in the corollary and the preceding discussion,

$$\begin{aligned} E \cap (A + A^\perp) &= (E \cap A) + (E \cap A^\perp) \\ &\stackrel{\phi}{\approx} (E' \cap A') + (E' \cap A'^\perp) = E' \cap (A' + A'^\perp). \end{aligned}$$

So (C2) holds. Moreover, the induced linear bijection ϕ_A in (17) is given by

$$\phi_A \left(\mathbf{v} + (A^\perp \cap A) \right) = \phi(\mathbf{v}) + (A^\perp \cap A), \quad \text{for } \mathbf{v} \in E \cap A,$$

which is an isometry. So Corollary 5.5 follows by Theorem 5.3. \square

REMARK 5.6. If an isometry $\phi_V : V \rightarrow V'$ sends E to E' and A to A' respectively, then

$$f(E, A) \approx \phi_V(f(E, A)) = f(\phi_V(E), \phi_V(A)) = f(E', A') \quad (66)$$

for every meaningful expression f that consists of brackets, $+$, \cap , and \perp (taking orthogonal complement). Nevertheless, the converse is not true! In the following counterexample, $E, A \subset V$ and $E', A' \subset V'$ satisfy that $f(E, A) \approx f(E', A')$ for every f defined above. However, there is no isometry $\phi_V : V \rightarrow V'$ that sends E to E' and A to A' respectively.

Suppose that $\text{char}(\mathbb{F}) = 0$. Let $V = V' = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ be the nonsingular symmetric metric space determined by $b(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}$ for $i, j = 1, 2$. Denote

$$A = A' := \mathbb{F}\mathbf{v}_1, \quad E := \mathbb{F}(\mathbf{v}_1 + 2\mathbf{v}_2), \quad E' := \mathbb{F}(\mathbf{v}_1 + 3\mathbf{v}_2).$$

Then

$$A^\perp = A'^\perp = \mathbb{F}\mathbf{v}_2, \quad E^\perp = \mathbb{F}(2\mathbf{v}_1 - \mathbf{v}_2), \quad E'^\perp = \mathbb{F}(3\mathbf{v}_1 - \mathbf{v}_2).$$

Denote a set \mathcal{S} (resp. \mathcal{S}') of subspaces of V (resp. V') as follow:

$$\mathcal{S} := \{\{\mathbf{0}\}, E, E^\perp, A, A^\perp, V\}, \quad \mathcal{S}' := \{\{\mathbf{0}'\}, E', E'^\perp, A', A'^\perp, V'\}.$$

It is easy to check that \mathcal{S} (resp. \mathcal{S}') is closed under the operations $+$, \cap , and \perp . Moreover, define a bijection $\varpi : \mathcal{S} \rightarrow \mathcal{S}'$ that sends $\{\mathbf{0}\}$ to $\{\mathbf{0}'\}$, E to E' , E^\perp to E'^\perp , A to A' , A^\perp to A'^\perp , V to V' , respectively. Then $X \approx \varpi(X)$ for every $X \in \mathcal{S}$, and ϖ commutes with $+$, \cap , and \perp operations. For every meaningful expression $f(X, Y)$ of subspaces X and Y in a metric space that consists of brackets, $+$, \cap , and \perp , we have $f(E, A) \in \mathcal{S}$ and

$$f(E, A) \approx \varpi(f(E, A)) = f(\varpi(E), \varpi(A)) = f(E', A').$$

However, there is no isometry $\phi_V : V \rightarrow V'$ that sends A to A' and E to E' simultaneously.

REMARK 5.7. The equality (C1) is implied by (C3) and (C4). However, if the specific isometry ϕ is removed from (C1)~(C4), then the resulting relations are independent of each other.

To take a glance, let $V = V' = \text{span}\{\mathbf{e}_1, \tilde{\mathbf{e}}_1, \mathbf{e}_2, \tilde{\mathbf{e}}_2, \mathbf{e}_3, \tilde{\mathbf{e}}_3\}$ be the nonsingular symmetric metric space defined by $b = b'$ and

$$b(\mathbf{e}_i, \mathbf{e}_j) = 0, \quad b(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) = 0, \quad b(\mathbf{e}_i, \tilde{\mathbf{e}}_j) = \delta_{ij}, \quad \text{for } i, j = 1, 2, 3.$$

Denote

$$E = E' := \text{span}\{\mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3\}, \quad A := \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \tilde{\mathbf{e}}_2\}, \quad A' := \text{span}\{\mathbf{e}_1, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 - \tilde{\mathbf{e}}_3\}.$$

Then

$$A^\perp = \text{span}\{\mathbf{e}_1, \mathbf{e}_3, \tilde{\mathbf{e}}_3\}, \quad A'^\perp = \text{span}\{\mathbf{e}_2 + \mathbf{e}_3, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}.$$

Evidently, $E \approx E'$, $A \approx A'$, $E \cap (A + A^\perp) \approx E' \cap (A' + A'^\perp)$, $E \cap A \approx E' \cap A'$, $E \cap A^\perp \approx E' \cap A'^\perp$, but $E \cap A^\perp \cap A$ and $E' \cap A'^\perp \cap A'$ are not isometric.

REMARK 5.8. Suppose $E, A \subset V$, $E', A' \subset V'$, and $\phi : E \rightarrow E'$ satisfy (C2)~(C4), so that ϕ_A and ϕ_{A^\perp} are well-defined. In the metric space $(\frac{A+A^\perp}{A^\perp \cap A}, \bar{b})$, the intersection of $\text{Dom}(\phi_A) = \frac{(E+A^\perp) \cap A}{A^\perp \cap A}$ and $\text{Dom}(\phi_{A^\perp}) = \frac{(E+A) \cap A^\perp}{A^\perp \cap A}$ is trivial, and $\text{Dom}(\phi_A) \perp \text{Dom}(\phi_{A^\perp})$. Likewise for $\text{Im}(\phi_A)$ and $\text{Im}(\phi_{A^\perp})$ in $(\frac{V'}{A'^\perp \cap A'}, \bar{b}')$. So we can define the linear bijection

$$\phi_A \odot \phi_{A^\perp} : \frac{[(E + A^\perp) \cap A] + [(E + A) \cap A^\perp]}{A^\perp \cap A} \rightarrow \frac{[(E' + A'^\perp) \cap A'] + [(E' + A') \cap A'^\perp]}{A'^\perp \cap A'}. \quad (67)$$

Then $\phi_A \odot \phi_{A^\perp}$ is an extension of the following isometry induced by ϕ :

$$\phi_{A+A^\perp} : \frac{[E \cap (A + A^\perp)] + (A^\perp \cap A)}{A^\perp \cap A} \rightarrow \frac{[E' \cap (A' + A'^\perp)] + (A'^\perp \cap A')}{A'^\perp \cap A'}, \quad (68)$$

where $\phi_{A+A^\perp}(\mathbf{e} + (A^\perp \cap A)) := \phi(\mathbf{e}) + (A'^\perp \cap A')$ for $\mathbf{e} \in E \cap (A + A^\perp)$.

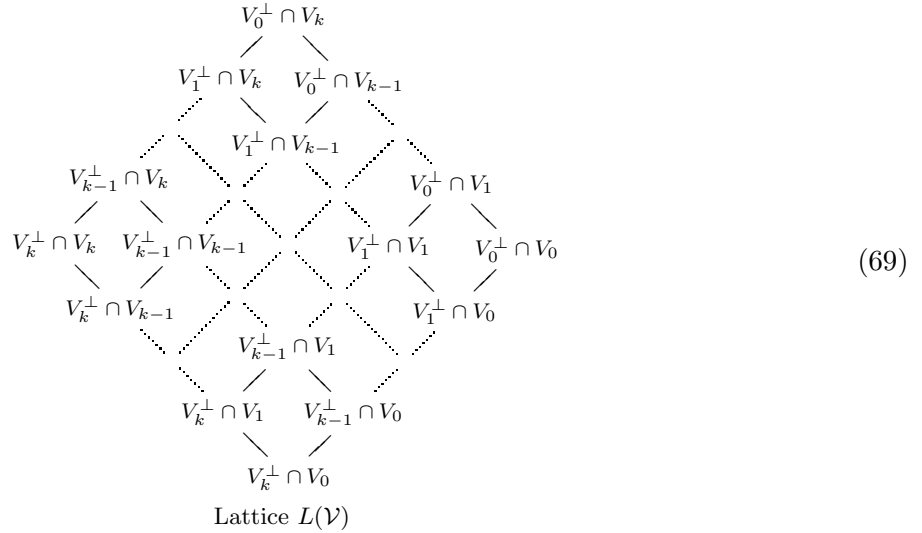
6. More about isometries over flags

We continue the study on isometry of flags in Section 3. First we give a criteria about the isometry of two generic flags. Then we determine when there exists an isometry that simultaneously sends a subspace to another, and a self-dual flag to another, respectively. The later object is analogous to Theorem 3.2 except that here no isometry ϕ is provided.

Let $V \approx V'$ be nonsingular isometric metric spaces. The following theorem is an extension of Lemma 2.4.

THEOREM 6.1. *Let $\mathcal{V} = \{V_i\}_{i=0, \dots, k}$ and $\mathcal{V}' = \{V'_i\}_{i=0, \dots, k}$ be generic flags of V and V' respectively. Then $\mathcal{V} \approx \mathcal{V}'$ if and only if $V_i \approx V'_i$ for $i = 0, \dots, k$ and $\dim(V_i^\perp \cap V_j) = \dim(V_i'^\perp \cap V'_j)$ for $0 < j < i < k$.*

Proof. We only need to prove the sufficient part.



The figure above displays a lattice $L(\mathcal{V})$ of subspaces in V , where the edges represent inclusion relations. The entries of $L(\mathcal{V})$ are $V_i^\perp \cap V_j$ where (i, j) is in the index set

$$I := \{0, \dots, k\} \times \{0, \dots, k\}. \quad (70)$$

Let us call $V_i^\perp \cap V_j$ the (i, j) -entry of $L(\mathcal{V})$. Of course, $V_k^\perp \cap V_j = V_i^\perp \cap V_0 = \{\mathbf{0}\}$, $V_0^\perp \cap V_j = V_j$ and $V_i^\perp \cap V_k = V_i^\perp$. Likewise, we denote a lattice $L(\mathcal{V}')$ with respect to \mathcal{V}' .

The subspaces in the “lower half part” of $L(\mathcal{V})$ (including those in the middle) are $V_i^\perp \cap V_j$ for $0 \leq j \leq i \leq k$, which are totally isotropic. Moreover, denote

$$T := \sum_{i=0}^k (V_i^\perp \cap V_i), \quad T' := \sum_{i=0}^k (V_i'^\perp \cap V_i'). \quad (71)$$

Then both T and T' are totally isotropic. By the sufficient conditions in the theorem, $\dim(V_i^\perp \cap V_j) = \dim(V_i'^\perp \cap V'_j)$ for $0 \leq j < i \leq k$, and $\dim(V_i^\perp \cap V_i) = \dim(V_i'^\perp \cap V_i')$ since $V_i \approx V'_i$. Using basis

extension and induction over the lattice entries of $L(\mathcal{V})$ and $L(\mathcal{V}')$, we can explicitly construct a linear bijection $\phi_0 : T \rightarrow T'$ such that $\phi_0(V_i^\perp \cap V_j) = V_i'^\perp \cap V_j'$ for $0 \leq j \leq i \leq k$. Then ϕ_0 is an isometry of totally isotropic subspaces.

Given $j \in \{0, \dots, k-1\}$, suppose $\phi_j : T + V_j \rightarrow T' + V_j'$ is an isometry such that $\phi_j|_T = \phi_0$ and $\phi_j(V_q) = V_q$ for $q = 0, \dots, j$. We show that ϕ_j can be extended to an isometry $\phi_{j+1} : T + V_{j+1} \rightarrow T' + V_{j+1}'$ such that $\phi_{j+1}(V_{j+1}) = V_{j+1}'$. This is done by applying Corollary 5.5 on $(V, b) \approx (V', b')$ with

$$E := T + V_j, \quad A := V_{j+1}, \quad E' := T' + V_j', \quad A' := V_{j+1}', \quad \text{and} \quad \phi := \phi_j. \quad (72)$$

Let us verify the sufficient conditions in Corollary 5.5:

(C3):

$$\begin{aligned} E \cap A &= \left[\sum_{i=0}^k (V_i^\perp \cap V_i) + V_j \right] \cap V_{j+1} = \left[\sum_{i=j+1}^k (V_i^\perp \cap V_i) + V_j \right] \cap V_{j+1} \\ &= \left(\left[\sum_{i=j+1}^k (V_i^\perp \cap V_i) \right] \cap V_{j+1} \right) + V_j \quad \text{by Lemma 2.1(5)} \\ &= \left(\left[\sum_{i=j+1}^k (V_i^\perp \cap V_i) \right] \cap V_{j+1}^\perp \cap V_{j+1} \right) + V_j \quad \text{by Lemma 2.1(6)} \\ &= (V_{j+1}^\perp \cap V_{j+1}) + V_j. \end{aligned}$$

Likewise, $E' \cap A' = (V_{j+1}'^\perp \cap V_{j+1}') + V_j'$. Then (C3) holds since

$$\phi \left((V_{j+1}^\perp \cap V_{j+1}) + V_j \right) = \phi_0(V_{j+1}^\perp \cap V_{j+1}) + \phi_j(V_j) = (V_{j+1}'^\perp \cap V_{j+1}') + V_j'.$$

(C4):

$$\begin{aligned} E \cap A^\perp &= \left[\sum_{i=0}^k (V_i^\perp \cap V_i) + V_j \right] \cap V_{j+1}^\perp = \left[\sum_{i=j+1}^k (V_i^\perp \cap V_i) + V_j \right] \cap V_{j+1}^\perp \\ &= \left[\sum_{i=j+1}^k (V_i^\perp \cap V_i) \right] + (V_{j+1}^\perp \cap V_j) \quad \text{by Lemma 2.1(5)} \\ &= \sum_{i=j+1}^k (V_i^\perp \cap V_i). \end{aligned}$$

Then $E \cap A^\perp \stackrel{\phi}{\approx} E' \cap A'^\perp$ by similar computation on $E' \cap A'^\perp$. Thus (C4) holds.

Besides these,

$$\begin{aligned}
 (E + A^\perp) \cap A &= \left[\sum_{i=0}^k (V_i^\perp \cap V_i) + V_j + V_{j+1}^\perp \right] \cap V_{j+1} \\
 &= \left[\sum_{i=j+1}^k (V_i^\perp \cap V_i) + V_j + V_{j+1}^\perp \right] \cap V_{j+1} \\
 &= (V_j + V_{j+1}^\perp) \cap V_{j+1} \\
 &= V_j + (V_{j+1}^\perp \cap V_{j+1}) \quad \text{by Lemma 2.1(5)} \\
 &= E \cap A.
 \end{aligned}$$

Similarly, $(E' + A'^\perp) \cap A' = E' \cap A'$. By Corollary 5.5, the isometry $\phi = \phi_j$ can be extended to an isometry $\tilde{\phi}_j : V \rightarrow V'$ that sends $A = V_{j+1}$ to $A' = V'_{j+1}$. Then the isometry $\phi_{j+1} := \tilde{\phi}_j|_{T+V_{j+1}}$ is an extension of ϕ_j that sends V_{j+1} to V'_{j+1} .

Finally, repeating the above process, we obtain a series of extension of isometries via $\phi_0 \rightarrow \phi_1 \rightarrow \dots \rightarrow \phi_k$. Notice that $T + V_k = V$ and $T' + V'_k = V'$. The isometry $\phi_k : V \rightarrow V'$ satisfies that $\phi_k(V_i) = V'_i$ for $i = 0, \dots, k$. So $\mathcal{V} \approx \mathcal{V}'$. \square

We return to the simultaneous isometry of (subspace, self-dual flag) pairs. As before, denote $E_i := E \cap V_i$ and $E'_i := E' \cap V'_i$. Recall that two self-dual flags $\mathcal{V} = \{V_i\}_{i=0, \dots, k}$ of V and $\mathcal{V}' = \{V'_i\}_{i=0, \dots, k}$ of V' are isometric if and only if $\dim V_i = \dim V'_i$ for $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$. The following result is analogous to Theorem 3.2.

THEOREM 6.2. *Suppose $E \subset V$ and $E' \subset V'$ where $V \approx V'$. Let $\mathcal{V} := \{V_i\}_{i=0, \dots, k}$ and $\mathcal{V}' := \{V'_i\}_{i=0, \dots, k}$ be isometric self-dual flags of V and V' respectively. Then there exists an isometry $\phi_V : V \rightarrow V'$ that sends E to E' and \mathcal{V} to \mathcal{V}' simultaneously if and only if $E_i \approx E'_i$ for $i = 1, \dots, k$, and $\dim(E_i^\perp \cap E_j) = \dim(E'_i{}^\perp \cap E'_j)$ for $k - i < j < i \leq k$.*

Proof. It suffices to prove the sufficient part. Note that $E_i^\perp \cap E_j = E_j$ whenever $i + j \leq k$. We have $\dim(E_i^\perp \cap E_j) = \dim(E'_i{}^\perp \cap E'_j)$ for $0 < j < i \leq k$. Consider the flags

$$\mathcal{E} := \{E_0 = \{\mathbf{0}\} \subset E_1 \subset \dots \subset E_k = E \subset V\} \quad \text{of} \quad V$$

and

$$\mathcal{E}' := \{E'_0 = \{\mathbf{0}'\} \subset E'_1 \subset \dots \subset E'_k = E' \subset V'\} \quad \text{of} \quad V'.$$

By Theorem 6.1, there is an isometry $\phi : V \rightarrow V'$ that sends \mathcal{E} to \mathcal{E}' , that is, $\phi(E_i) = E'_i$ for $i = 0, \dots, k$. By Theorem 3.2, the isometry $\phi|_E : E \rightarrow E'$ can be extended to an isometry $\phi_V : V \rightarrow V'$ that sends \mathcal{V} to \mathcal{V}' . This completes the proof. \square

COROLLARY 6.3. *Suppose $V \approx V'$. Suppose $E, A \subset V$ and $E', A' \subset V'$ where $A \approx A'$ are totally isotropic and $E \approx E'$. Then there exists an isometry $\phi_V : V \rightarrow V'$ that sends E to E' and A to A' simultaneously if and only if*

$$E \cap A^\perp \approx E' \cap A'^\perp, \tag{73a}$$

$$\dim(E \cap A) = \dim(E' \cap A'), \tag{73b}$$

$$\dim(E^\perp \cap E \cap A) = \dim(E'^\perp \cap E' \cap A'), \tag{73c}$$

$$\dim(E^\perp \cap E \cap A^\perp) = \dim(E'^\perp \cap E' \cap A'^\perp). \tag{73d}$$

Proof. Denote the self-dual flags $\mathcal{V} := \{\{\mathbf{0}\} \subset A \subset A^\perp \subset V\}$ and $\mathcal{V}' := \{\{\mathbf{0}'\} \subset A' \subset A'^\perp \subset V'\}$. Then apply Theorem 6.2 to get the result. \square

If each of E , A , E^\perp , and A^\perp in Corollary 6.3 is not totally isotropic, then by Remark 5.6, we may not be able to determine the simultaneous isometry of (E, A) pair with (E', A') pair by solely inspecting the isometries of the corresponding subspaces related to the pairs via intersections, additions, and taking orthogonal complements.

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Huajun Huang huanghu@auburn.edu

Department of Mathematics and Statistics, Auburn University, AL 36849-5310, USA